Universidad Autónoma de San Luis Potosí.

Facultad de Ciencias

On The Structural Stability Of Weakly Coupled Map Lattices

TESIS

Que para obtener el Grado de

Maestro en Ciencias Aplicadas

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San Luis Potosí, S.L.P. Noviembre de 2003.

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A Mi Familia

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Acknowledgements

I would like to thank Ph.D. Valentin Afraimovich and Ph.D. Antonio Morante, my advisors, for their suggestions and constant support during this work.

I am grateful with Ph.D. Gelasio Salazar, Ph.D. Edgardo Ugalde and Ph.D. Jesús Urias, they are wonderful people.

Of course, I am grateful with my family for their love, they are my principal support. Without them nothing had been come into existence.

Thanks to Norma L. Carranco for change my life.

Finally, thanks to: Jesús Leaños, Luis Reyna, Jorge Ortega, Carlos Jacob Rubio and Efrain Castillo for their friendship; thanks to the people that helped me along my life.

I would like to thank to CONACYT for its economic support during my studies in IICO–UASLP.

Preface

Coupled Map Lattices (CML) are infinite dimensional dynamical systems with discrete spatial and temporal variables, which are used as models of spatially extended non equilibrium media (see for instance [7], [8]). They can be considered as discrete versions of partial differential equations.

Although there are some results related to ergodic properties of CML with space–time chaos (see [9], [6]), there is no general theory of the topological properties of such CML. Some results about the topological behavior of 1-dimensional homogeneous CML can be found in [1], [2] and [5].

The goal of our work is to prove some kind of structural stability of Weakly Coupled Map Lattices for the d-dimensional non homogeneous case. In order to do that, we show that there is a conjugacy (in the product topologies) between the uncoupled system and the slightly perturbed one, constructing symbolic representations of weakly CML via topological Markov chains.

The present work is divided as follows:

In Chapter 1 we introduce a general class of extended dynamical systems, the so called Lattice Dynamical Systems (**LDS**), showing particular examples of them. Then we give a definition of LDS including symbolic systems.

In Chapter 2 we study the conjugacy of the local maps with topological Markov chains. Then we define the uncoupled system, showing the existence of the symbolic representation for this system (which is, in fact, the direct product of the local symbolic systems). Finally, we introduce near neighbors interaction type CML (being the discrete version of the reaction–diffusion equation the most popular of them).

In Chapter 3 we study near neighbors interaction type CML as perturbations of the uncoupled subsystems, proving the persistence of their dynamics whenever the coupling is weak enough and the local maps involved satisfy some conditions of hyperbolicity.

Such type of results have been known for the case of identical individual maps (see for instance [2]). The novelty of this work is, mainly, in consideration of non–homogeneous situation. Moreover, in this work we consider d–dimensional LDS, $d \geq 1$.

Chapter 1

Lattice Dynamical Systems

1.1 Introduction

The basic goal of the theory of discrete time dynamical systems is to understand the long term behavior of an iterative process. When the process is modeled by the iteration of a function, we want to describe the asymptotic behavior of the points x, $f(x)$, $f^2(x)$,..., $f^n(x)$ as n becomes large. So, the question is, in the long run, where do points go and what do they do when they get there? The answer is know; the points tend to invariant sets consisting of non–wandering orbits. Thus the problem of description of invariant non–wandering sets, appear to be very important.

Here we consider such problem for a particular class of Dynamical Systems, the so called Lattice Dynamical Systems (**LDS**). They are infinite dimensional dynamical systems belonging to a class of models of spatially extended media in which the relations between temporal evolution and spatial translation play an important role. The invariant with respect to spatial translations LDS are similar to autonomous Partial Differential Equations (**PDE**), non–homogeneous LDS are similar to PDE with coefficients depending on space variables.

The most common class of LDS is the so called Coupled Map Lattices (**CML**). A natural source of CML are discrete versions of partial differential equations of evolution type, which arise while modeling PDE's by computer.

Sometimes the use of lattice may be regarded as an approximate approach to description of a continuous medium, then the equations should have a reasonable continuous limit with decrease of the spatial step. In other cases, the lattice model may be appropriate essence of the problem. For example, in solid state physics a natural discretization appears due to presence of crystal lattice.

In the present work we study some properties of LDS. As the main result we show the structural stability of non–coupled hyperbolic maps. The basic idea behind this calculations is to take advantage of symbolic dynamics once we have showed the existence of a symbolic representation for CML with a weak coupling.

1.2 Basic notions

Lattice Dynamical Systems occur in a wide variety of applications where the spatial structure has a discrete character. We begin with a definition of LDS.

Definition 1 *Let I be a (subset of a) metric space and let $\rho(\cdot,\cdot)$ be the corresponding distance. Consider the direct product $I^{\mathbb{Z}^d}$ endowed with the uniform or with the product topology. Assume the existence of a subset $\mathcal{M} \subset I^{\mathbb{Z}^d}$ and the existence of a map $\mathcal{F}$ from $\mathcal{M}$ into itself. The pair $(\mathcal{M}, \mathcal{F})$ is called a d–dimensional* **lattice dynamical system**.

An **orbit** *of $\mathcal{F}$ in $\mathcal{M}$ is a sequence $\{u(t)\}_{t\in\mathbb{N}}$ where $u(t) = \{u_s(t)\}_{s\in\mathbb{Z}^d}$ belongs to $\mathcal{M}$ and $u(t+1) = \mathcal{F}(u(t))$ for all $t \in \mathbb{N} \doteq \{0, 1, 2, \ldots\}$.*

From now on, we suppose that the set $\mathcal{M}$ is compact and the map $\mathcal{F}$ is continuous in the product topology.

Some examples of LDS are the following

1. Cellular Automata[1](CA) for which I is a finite alphabet endowed with the discrete topology and the evolution map $\mathcal{F}$ is a continuous map which commutes with spatial translations

2. Near neighbors interaction type CML with diffusive coupling, for which $\mathcal{M} \subset I^{\mathbb{Z}}$, I is a compact interval and the evolution map $\mathcal{F}$ is given by

$$(\mathcal{F}u)_s = f_s(u_s) + \gamma F(u_{s-r}, \ldots, u_{s+r}),$$

for $u = \{u_s\}_{s \in \mathbb{Z}} \in I^{\mathbb{Z}}$. Here $\gamma > 0$ is the coupling parameter; the local maps $f_s : \mathbb{R} \longrightarrow \mathbb{R}$ and the coupling maps $F_s : \mathbb{R}^{2r+1} \longrightarrow \mathbb{R}$ are C^1–smooth. In this model the dynamics consists of two independent components: the local (individual) dynamics and the coupling dynamics. The former one is the application of the one dimensional *local map* f_s to every site u_s and the latter one couples the dynamics by means of a weighted map F_s over a neighborhood of u_s where the coefficient γ determine the size of coupling interaction.

[1]CA were the first LDS that attracted considerable interest. CML where introduced by Kaneko, K. [1983,1984] as a model for the study of spatio–temporal complexity such as turbulence, population dynamics, etc. We could think of CML as a generalization of CA.

Chapter 2

Coupled Map Lattices

2.1 Local maps

In this work we consider non–homogeneous d–dimensional CML.

Let $\{I_s\}_{s\in\mathbb{Z}^d}$ be a family of closed subintervals of the interval $I = [a, b]$. Let us suppose that, for each $s \in \mathbb{Z}^d$, $f_s : I_s \to \mathbb{R}$ is a C^1–smooth map satisfying the following chaotic hypothesis:

(H1) there exist a finite collection of pairwise disjoint closed subintervals $\{I_s^i\}_{i=1}^{p_s}$ of $I_s \subset I$ such that if $1 \leq i \leq p_s$:

a. f_s is differentiable on I_s^i with $1 < \alpha \doteq \inf_{s\in\mathbb{Z}^d} \left\{ \min_{x\in I_s^i} |f_s'(x)| \right\} < +\infty$,

b. there exists $1 \leq j \leq p_s$ such that $I_s^j \subset \text{Int} f_s(I_s^i)$.

From now on we suppose that, for each $s \in \mathbb{Z}^d$, $p_s < p < \infty$.

As an example of a map similar to the local maps used, consider the family of quadratic maps $F_\mu(x) = \mu x(1 - x)$ with $\mu > 2 + \sqrt{5}$. Make $C = I \setminus A$, where $I = [0, 1]$ and $A = \{x \in I : |f(x)| > 1\}$.

A direct calculation shows that if $\mu > 4$ then the local extreme of f is, bigger than 1, moreover if $\mu > 2 + \sqrt{5}$ we have that $|f'(x)| > 1$ on $C = I_1 \cup I_2$ with

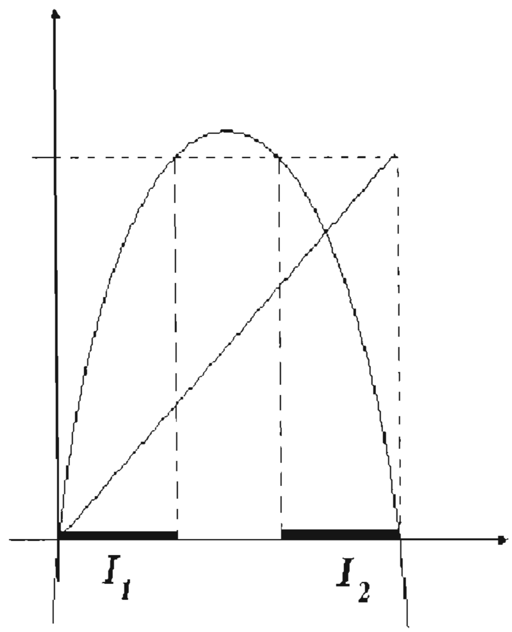

Figure 2.1: The quadratic non–linearity $f(x) = \mu x(1-x)$.

$$I_1 = \left[0, \frac{1}{2} - \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}\right] \quad ; \quad I_2 = \left[\frac{1}{2} + \frac{\sqrt{\mu^2 - 4\mu}}{2\mu}, 1\right]$$

Note the existence of a closed and f–invariant subset Λ_f of $C \subset I$ on which f is topologically conjugate to the full shift in 2 symbols. (see [4]).

In the general case, if the local map f_s satisfies H1, there exists a closed and f_s–invariant subset Λ_{f_s} of $C_s = \bigcup_{i=1}^{p_s} I_s^i \subset I_s$, on which (Λ_{f_s}, f_s) is topologically conjugate to a topological Markov chain (Ω_{A_s}, σ). Here σ is the shift map on the set of admissible (with respect to the transition matrix A_s) sequences in $\Omega_{A_s} \subset \{1, \cdots, p_s\}^{\mathbb{N}}$, endowed with the metric

$$d(w, \bar{w}) = \exp(-k); \quad w = \{w^t\}_{t \in \mathbb{N}}, \bar{w} = \{\bar{w}^t\}_{t \in \mathbb{N}} \in \Omega_{A_s}$$

with $k = \min\{t \in \mathbb{N} : w^t \neq \bar{w}^t\}$.

In other words, there exists a homeomorphism $\pi_s : \Omega_{A_s} \to \Lambda_{f_s}$, (respect to the previous and the Euclidean metrics) such that

$$\pi_s \circ \sigma = f_s \circ \pi_s.$$

2.2 Uncoupled systems

An important concept is that of structural stability; some types of systems have dynamics which are equivalent (topologically conjugated) to that of any of its perturbations. Our goal is to proof the structural stability of the d–dimensional lattices of uncoupled hyperbolic maps (see Chapter 3)

Given a family $\{f_s\}_{s\in\mathbb{Z}^d}$ of maps satisfying H1, let us make $\Lambda_0 \doteq \bigotimes_{s\in\mathbb{Z}^d} \Lambda_{f_s}$ and $\Sigma \doteq \bigotimes_{s\in\mathbb{Z}^d} \Omega_{A_s}$.

It is simple to see that the uncoupled map $\mathcal{F}_0 : \Lambda_0 \to \Lambda_0$ given by

$$(\mathcal{F}_0\, u)_s = f_s(u_s), \quad u = \{u_s\}_{s\in\mathbb{Z}^d} \tag{2.1}$$

is topologically conjugate to the time–translation operator $\sigma_\tau : \Sigma \to \Sigma$ defined by

$$(\sigma_\tau \omega)_s^t = \omega_s^{t+1}, \quad \omega = \{\omega_s^t\}_{s\in\mathbb{Z}^d, t\in\mathbb{N}} \in \Sigma.$$

Indeed, the map $\Pi_0 : \Sigma \to \Lambda_0$ defined by $(\Pi_0\, \omega)_s = \pi_s\, \omega_s$ is a homeomorphism (when we consider the systems endowed with the product topology) such that

$$\Pi_0 \circ \sigma_\tau = \mathcal{F}_0 \circ \Pi_0.$$

In the next section we present Coupled Map Lattices $(\mathcal{M}, \mathcal{F})$ that are still conjugated to the symbolic system (Σ, σ_τ) provided that $\mathcal{F}$ is a small perturbation of $\mathcal{F}_0$.

2.3 Weakly coupled systems

Here we consider perturbations of the uncoupled system 2.1. Explicitly, define the map $\mathcal{F} : \bigotimes_{s \in \mathbb{Z}^d} I_s \to \mathbb{R}^{\mathbb{Z}^d}$ such that for each $s = (s_1, \ldots s_d) \in \mathbb{Z}^d$,

$$(\mathcal{F}u)_s = f_s(u_s) + \gamma F_s(\langle u_s \rangle^r) \tag{2.2}$$

where

$$\langle u_s \rangle^r \doteq \left(u_{(s_1-r,\ldots,s_d-r)}, \ldots, u_{(s_1-r,\ldots,s_d+r)}, \ldots, u_{(s_1+r,\ldots,s_d-r)}, \ldots, u_{(s_1+r,\ldots,s_d+r)}\right)$$

is the $(2r+1)^d$–tuple of neighbors of u_s (ordered in the dictionary order).

From now on, we suppose that

- the local maps $f_s : I_s \to \mathbb{R}$ are C^1–smooth maps satisfying the hypothesis H1 (see Section 2.1),
- the coupling maps $F_s : \mathbb{R}^{(2r+1)^d} \to \mathbb{R}$ are C^1–smooth with bounded first partial derivatives. Explicitly,

$$\beta \doteq \sup_{s \in \mathbb{Z}^d} \left\{ \max_{1 \le j \le (2r+1)^d} \left| \frac{\partial F_s}{\partial x_j} \left(x_1, \ldots, x_{(2r+1)^d}\right) \right| \right\} < +\infty,$$

  where the maximum is taken for all the possible choices of $\left(x_1, \ldots, x_{(2r+1)^d}\right)$ in $\langle C_s \rangle^r \doteq C_{(s_1-r,\ldots,s_d-r)} \times \ldots \times C_{(s_1-r,\ldots,s_d+r)} \times \ldots \times C_{(s_1+r,\ldots,s_d-r)} \times \ldots \times C_{(s_1+r,\ldots,s_d+r)}$,
- $0 < \delta \doteq \inf_{s \in \mathbb{Z}^d} \left\{ \min\{a_s^j - m_s^i, M_s^i - b_s^j\} \right\}$, where the minimum is taken over all the pairs $1 \le i, j \le p_s$, $s \in \mathbb{Z}^d$, such that

$$I_s^j \doteq [a_s^j, b_s^j] \subset \operatorname{Int} f_s(I_s^i) \doteq (m_s^i, M_s^i).$$

Chapter 3

Structural stability of the uncoupled system $\mathcal{F}_0$

3.1 Main theorem

The following result gives us a symbolic description in $\Sigma = \bigotimes_{s \in \mathbb{Z}^d} \Omega_{A_s}$ of all orbits of the system 2.2 in the set $\mathcal{C} = \bigotimes_{s \in \mathbb{Z}^d} C_s$ (see Sections 2.1 and 2.3).

Theorem 1 *There exists $\gamma_{\mathcal{F}} > 0$ such that, for any coupling parameter $0 \leq \gamma < \gamma_{\mathcal{F}}$, there is an $\mathcal{F}$-invariant closed set $\Lambda_{\mathcal{F}} \subset \mathcal{C}$ and a bijective map $\Pi : \Sigma \to \Lambda_{\mathcal{F}}$ so that*

$$\Pi \circ \sigma_\tau = \mathcal{F} \circ \Pi \tag{3.1}$$

when $\mathcal{F}$ is restricted to $\Lambda_{\mathcal{F}}$.

The map $\Pi : \Sigma \to \Lambda_{\mathcal{F}}$ is a homeomorphism in the product topologies.

Proof. To make the proof more simple, we provide Σ and $\mathcal{C}$ with the following distances compatible with product topologies. For $s = (s_1, \ldots, s_d) \in \mathbb{Z}^d$ we use the norm

$$\| s \| = \max\{|s_1|, \ldots, |s_d|\}.$$

Fix a number $q > 1$, the distance on Σ is given by

$$D_q(\omega, \bar{\omega}) = \sum_{s \in \mathbb{Z}^d} q^{-\| s \|} \, d(\omega_s, \bar{\omega}_s),$$

and the distance on C is given by

$$\|u - \bar{u}\|_q = \sum_{s\in\mathbb{Z}^d} q^{-\|s\|} |u_s - \bar{u}_s|.$$

We prove now a generalization of the property H1.b for $\mathcal{F}$, provided the coupling is weak enough.

Lemma 1 *There exists $\gamma_1 > 0$ such that, if $w = \{w_s\}_{s\in\mathbb{Z}^d}$ and $w' = \{w'_s\}_{s\in\mathbb{Z}^d}$ are infinite labels in $\bigotimes_{s\in\mathbb{Z}^d}\{1,\ldots,p_s\}$ for which $I_s^{w'_s} \subset \mathrm{Int}(f_s(I_s^{w_s}))$ for all $s \in \mathbb{Z}^d$, then for $0 \le \gamma \le \gamma_1$ in (2.2) we have*

$$I^{w'} \subset \mathcal{F}(I^w) \qquad \text{where} \qquad I^w = \bigotimes_{s\in\mathbb{Z}^d} I_s^{w_s} \quad \text{and} \quad I^{w'} = \bigotimes_{s\in\mathbb{Z}^d} I_s^{w'_s}.$$

Proof. Let w, w' infinite labels like in the statement of the lemma. By hypothesis

$$m_s^{w_s} < a_s^{w'_s} \le b_s^{w'_s} < M_s^{w_s}.$$

Make $M \doteq \sup_{s\in\mathbb{Z}^d}\left\{\ \max |F_s\langle x_1,\ldots,x_{(2r+1)^d}\rangle|\ \right\}$, where the maximum is taken for all the possible choices of $\langle x_1,\ldots,x_{(2r+1)^d}\rangle \in I^{(2r+1)^d}$. For each $s \in \mathbb{Z}^d$ let $\underline{u}_s, \bar{u}_s \in I_s^{w_s}$ such that $f_s(\underline{u}_s) = m_s^{w_s}$ and $f_s(\bar{u}_s) = M_s^{w_s}$. Take $\gamma_1 = \dfrac{\delta}{2M}$. Then, for each $s \in \mathbb{Z}^d$ we have,

$$f_s(\underline{u}_s) + \gamma_1 \,|\, F_s(\langle \underline{u}_s\rangle^r)| < f_s(\underline{u}_s) + \delta \le a_s^{w'_s}$$

and

$$b_s^{w'_s} \le f_s(\bar{u}_s) - \delta < f_s(\bar{u}_s) - \gamma_1 \,|\, F_s(\langle \bar{u}_s\rangle^r)|.$$

The continuity of f_s and F_s implies that $I^{w'_s} \subset \mathrm{proj}_s(\mathcal{F}(I^w))$, where $\mathrm{proj}_s : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ is the projection over the coordinate $s = (s_1, \ldots, s_d)$. Since this is true for all $s \in \mathbb{Z}^d$, $I^{w'} \subset \mathcal{F}(I^w)$. □

Lemma 2 *There exists γ_2 such that, if the coupling parameter $0 \leq \gamma < \gamma_2$, then for each sequence $w = \{w(k)\}_{k\in\mathbb{N}}$ of infinite labels $\{w_s(k)\}_{s\in\mathbb{Z}^d}$ in $\bigotimes_{s\in\mathbb{Z}^d}\{1,\ldots,p_s\}$, the set*

$$\bigcap_{k\in\mathbb{N}} \mathcal{F}^{-k}(I^{w(k)})$$

contains exactly one element, whenever $w = \{w_s^k\}_{s\in\mathbb{Z}^d, k\in\mathbb{N}} \in \Sigma$. (here $I^{w(k)} = \bigotimes_{s\in\mathbb{Z}^d} I_s^{w_s(k)}$).

Proof: For $0 \leq \gamma < \gamma_1$ given, fix $w \in \Sigma$. Since $I^{w(k+1)} \subset \mathcal{F}(I^{w(k)})$, we have that

$$J_{w(0)\cdots w(k)} \doteq \bigcap_{t=0}^{k} \mathcal{F}^{-t}(I^{w(t)}) \neq \emptyset.$$

These sets are closed and form a nested sequence; furthermore, the J's sets of the same generation are pairwise disjoint.

Make

$$\operatorname{diam}(J_{w(0)\cdots w(k)}) \doteq \sup_{s\in\mathbb{Z}^d} \{\operatorname{diam}(\operatorname{proj}_s(J_{w(0)\cdots w(k)}))\}.$$

We show that

$$\lim_{k\to\infty} \operatorname{diam}(J_{w(0)\cdots w(k)}) = 0. \tag{3.2}$$

If $\bar{u}, \bar{v} \in I^{w(1)}$, then $\bar{u}, \bar{v} \in \mathcal{F}(I^{w(0)})$; that is, $\bar{u} = \mathcal{F}u$ and $\bar{v} = \mathcal{F}v$ for some $u, v \in I^{w(0)}$. Using the mean value theorem we can write, for suitable $y_s \in I_s^{w_s(0)}$ and $(z_s^1, \ldots, z_s^{(2r+1)^d}) \in \langle I_s^{w_s(0)} \rangle^r$,

$$(\mathcal{F}u)_s - (\mathcal{F}v)_s = f_s'(y_s)\,(u_s - v_s) + \gamma DF_s(z_s^1, \ldots, z_s^{(2r+1)^d}) \cdot (u_s - v_s)$$

or

$$f_s'(y_s)(u_s - v_s) = (\bar{u} - \bar{v})_s - \gamma DF_s(z_s^1, \ldots, z_s^{(2r+1)^d}) \cdot (u - v)_s.$$

From the definition of α and β it follows that

$$\alpha \| u - v\|_\infty \leq \| \bar{u} - \bar{v}\|_\infty + \gamma\beta(2r+1)^d \| u - v\|_\infty$$

or

$$\| u - v\|_\infty \leq [\alpha - \gamma\beta(2r+1)^d]^{-1} \| \bar{u} - \bar{v}\|_\infty .$$

Therefore

$$\operatorname{diam} J_{w(0)w(1)} = \operatorname{diam}(I^{w(0)} \cap \mathcal{F}^{-1}(I^{w(1)})) \leq [\alpha - \gamma\beta(2r+1)^d]^{-1} \operatorname{diam}(I^{w(1)}).$$

Similarly, for $\bar{\bar{u}}, \bar{\bar{v}} \in I^{w(2)}$ let $\bar{u}, \bar{v} \in I^{w(1)}$ and $u, v \in I^{w(0)}$ such that $\mathcal{F}u = \bar{u}$, $\mathcal{F}v = \bar{v}$, $\mathcal{F}\bar{u} = \bar{\bar{u}}$ and $\mathcal{F}\bar{v} = \bar{\bar{v}}$. Then,

$$\| u - v\|_\infty \leq [\alpha - \gamma\beta(2r+1)^d]^{-2} \| \bar{\bar{u}} - \bar{\bar{v}}\|_\infty$$

and consequently

$$\begin{aligned} \operatorname{diam}(J_{w(0)w(1)w(2)}) &= \operatorname{diam}\left(I^{w(0)} \cap \mathcal{F}^{-1}(I^{w(1)}) \cap \mathcal{F}^{-2}(I^{w(2)})\right) \\ &\leq [\alpha - \gamma\beta(2r+1)^d]^{-2} \operatorname{diam}\left(I^{w(2)}\right) \end{aligned}$$

Inductively we obtain

$$\operatorname{diam} J_{w(0)w(1)\ldots w(k)} \leq [\alpha - \gamma\beta(2r+1)^d]^{-k} \operatorname{diam}(I^{w(k)}).$$

If $\gamma < \gamma_2 \doteq \min\left\{\gamma_1, \dfrac{\alpha - 1}{\beta(2r+1)^d}\right\}$, the limit (3.2) holds. □

We continue with the proof of Theorem 1. Given $w = \{w_s^k\}_{s\in\mathbb{Z}^d, k\in\mathbb{N}}$ in Σ, make

$$\Pi(w) = \bigcap_{k\in\mathbb{N}} \mathcal{F}^{-k}(I^{w(k)}).$$

This defines the map $\Pi : \Sigma \longrightarrow \Pi(\Sigma) \doteq \Lambda_{\mathcal{F}}$. Evidently, the set $\Lambda_{\mathcal{F}}$ is closed. Moreover, since the J's sets of the same generation are closed and disjoint, $\Lambda_{\mathcal{F}}$ is totally disconnected and Π is one–to–one.

Let us note that for $\omega = \{\omega_{\mathbf{s}}^k\}_{\mathbf{s}\in\mathbb{Z}^d,\, k\in\mathbb{N}} \in \Sigma$, since $I^{\omega(1)} \subset \mathcal{F}(I^{\omega(0)})$ and $\bigcap_{k\in\mathbb{Z}^+} \mathcal{F}^{-k+1}(I^{\omega(k)})$ is a singleton, we have

$$\begin{aligned}
\mathcal{F}\left(\bigcap_{k\in\mathbb{N}} \mathcal{F}^{-k}(I^{\omega(k)})\right) &= \mathcal{F}\big((I^{\omega(0)}) \cap \mathcal{F}^{-1}(I^{\omega(1)}) \cap \ldots \cap \mathcal{F}^{-k}(I^{\omega(k)}) \cap \ldots\big) \\
&= \mathcal{F}(I^{\omega(0)}) \cap (I^{\omega(1)}) \cap \ldots \cap \mathcal{F}^{-k+1}(I^{\omega(k)}) \cap \ldots \\
&= \bigcap_{k\in\mathbb{Z}^+} \mathcal{F}^{-k+1}(I^{\omega(k)}),
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Pi(\sigma_\tau(\omega)) &= \Pi\big(\{\omega_{\mathbf{s}}^{k+1}\}_{\mathbf{s}\in\mathbb{Z}^d,\, k\in\mathbb{N}}\big) \\
&= \bigcap_{k\in\mathbb{N}} \mathcal{F}^{-k}(I^{\omega(k+1)}) \\
&= \bigcap_{k\in\mathbb{Z}^+} \mathcal{F}^{-k+1}(I^{\omega(k)}) \\
&= \mathcal{F}\left(\bigcap_{k\in\mathbb{N}} \mathcal{F}^{-k}(I^{\omega(k)})\right) \\
&= \mathcal{F}(\Pi(\omega)).
\end{aligned}$$

Finally we show that Π is a homeomorphism in the product topologies. Because of compactness of Σ in the product topology, and since Π is one-to-one, we only have to prove that Π is continuous.

In what follows we use the ceiling function $\lceil\cdot\rceil$ (i.e. $\lceil x\rceil \in \mathbb{Z}$, $x \le \lceil x\rceil < x+1$).

Let $M \doteq min\left\{2, \frac{\alpha+1}{2}\right\}$ and $\mathcal{D} = \sup_{\mathbf{s}\in\mathbb{Z}^d}\big(\max_{1\le i\le p_{\mathbf{s}}} diam(I_{\mathbf{s}}^i)\big)$.

Lemma 3 *For any* $0 < \hat{\varepsilon} < min\{1, \mathcal{D}/M\}$*, make* $n = \left\lceil \dfrac{\log \mathcal{D} - \log \hat{\varepsilon}}{\log M} \right\rceil$. *If* $\omega = \{\omega_{\mathrm{s}}^t\}_{\mathrm{s}\in\mathbb{Z}^d, t\in\mathbb{N}}$, $\bar{\omega} = \{\bar{\omega}_{\mathrm{s}}^t\}_{\mathrm{s}\in\mathbb{Z}^d, t\in\mathbb{N}}$ *are configurations in* Σ *satisfying*

$$\omega_{\mathrm{s}}^t = \bar{\omega}_{\mathrm{s}}^t; \quad \| \mathrm{s} \| \leq nr, \quad 0 \leq t < n$$

then

$$|(\Pi\omega)_0 - (\Pi\bar{\omega})_0| < \hat{\varepsilon}.$$

Corollary 1 *For* $j \geq 0$ *given, if* $\omega = \{\omega_{\mathrm{s}}^t\}_{\mathrm{s}\in\mathbb{Z}^d,\ t\in\mathbb{N}}$ *and* $\bar{\omega} = \{\bar{\omega}_{\mathrm{s}}^t\}_{\mathrm{s}\in\mathbb{Z}^d,\ t\in\mathbb{N}}$ *are configurations in* Σ *satisfying that*

$$\omega_{\mathrm{s}}^t = \bar{\omega}_{\mathrm{s}}^t; \quad \| \mathrm{s} \| \leq nr + j, \quad 0 \leq t < n$$

then

$$| (\Pi\omega)_{\mathrm{s}} - (\Pi\bar{\omega})_{\mathrm{s}} | < \hat{\varepsilon}, \quad \| \mathrm{s} \| \leq j.$$

We prove the contrapositive of Lemma 3, that is, if $| (\Pi\omega)_0 - (\Pi\bar{\omega})_0 | \geq \hat{\varepsilon}$ then $\omega_{\mathrm{s}}^t \neq \bar{\omega}_{\mathrm{s}}^t$ for some $\mathrm{s} \in \mathbb{Z}^d$, $t \in \mathbb{N}$ satisfying $\| \mathrm{s} \| \leq nr$ and $0 \leq t < n$.

Proof. Let $u \doteq \Pi\omega$ and $\bar{u} \doteq \Pi\bar{\omega}$ in $\Lambda_{\mathcal{F}}$ and suppose that

$$| u_0 - \bar{u}_0 | \geq \hat{\varepsilon}.$$

If the points u_0 and $\bar{u}_0$ already lie in different subintervals of I, the configurations w_0 and $\bar{w}_0$ must differ in their first symbol, that is

$$\omega_0^0 \neq \bar{\omega}_0^0$$

If it is not so, making $u_{\mathrm{s}_0}(0) = u_0$ and $\bar{u}_{\mathrm{s}_0}(0) = \bar{u}_0$ we will construct a sequence $\{(\mathrm{s}_i\ ,\ t_i)\}_{i=1}^n \in \mathbb{Z}^d \times \mathbb{N}$, with $n = \left\lceil \dfrac{\log \mathcal{D} - \log \hat{\varepsilon}}{\log M} \right\rceil$, such that

$$|u_{\mathrm{s}_{i+1}}(t_{i+1}) - \bar{u}_{\mathrm{s}_{i+1}}(t_{i+1})| \geq M|u_{\mathrm{s}_i}(t_i) - \bar{u}_{\mathrm{s}_i}(t_i)|, \quad 0 \leq i \leq n-1.$$

Given i, $0 \leq i \leq n-1$ consider the quantity

$$\delta_i \doteq \max_{\| j \| \leq r} \{ |u_{s_i+j}(t_i) - \bar{u}_{s_i+j}(t_i)| \}.$$

If $\delta_i > 2|u_{s_i}(t_i) - \bar{u}_{s_i}(t_i)|$, we make $t_{i+1} = t_i$, choosing $s_{i+1} \in \langle u_s \rangle^r$ such that

$$|u_{s_{i+1}}(t_i) - \bar{u}_{s_{i+1}}(t_i)| = \delta_i.$$

So,

$$| u_{s_{i+1}}(t_{i+1}) - \bar{u}_{s_{i+1}}(t_{i+1}) | \geq 2 \, | u_{s_i}(t_i) - \bar{u}_{s_i}(t_i) |$$

if $\delta_i \leq 2 \, | u_{s_i}(t_i) - \bar{u}_{s_i}(t_i) |$, we let the system evolve, writing for suitable $y_i \in I_{s_i}$ and $\langle z_i^1, \ldots, z_i^{(2r+1)^d} \rangle \in \bigotimes_{\| j \| \leq r} I_{s_i+j}$,

$$\begin{aligned}
&\left| u_{s_i}(t_i+1) - \bar{u}_{s_i}(t_i+1) \right| \\
&= \left| f'_{s_i}(y_i) \left(u_{s_i}(t_i) - \bar{u}_{s_i}(t_i) \right) + \gamma \sum_{\| j \| \leq r} DF_{s_i}(z^1_{s_i}, \ldots, z^{(2r+1)^d}_{s_i}) \left(u_{s_i+j}(t_i) - u_{s_i+j}(t_i) \right) \right| \\
&\geq \left| f'_{s_i}(y_i) \right| \left| u_{s_i}(t_i) - \bar{u}_{s_i}(t_i) \right| - \gamma \sum_{\| j \| \leq r} \left| DF_{s_i}(z^1_{s_i}, \ldots, z^{(2r+1)^d}_{s_i}) \right| \left| u_{s_i+j}(t_i) - \bar{u}_{s_i+j}(t_i) \right| \\
&\geq \left[\alpha - 2\gamma\beta(2r+1)^d \right] \left| u_{s_i}(t_i) - \bar{u}_{s_i}(t_i) \right|.
\end{aligned}$$

Let $\gamma_{\mathcal{F}} \doteq \min \left\{ \gamma_1, \dfrac{\alpha - 1}{4\beta(2r+1)^d} \right\}$. If $0 < \gamma < \gamma_{\mathcal{F}}$, the condition over γ yields to

$$|u_{s_{i+1}}(t_{i+1}) - \bar{u}_{s_{i+1}}(t_{i+1})| \geq \frac{\alpha + 1}{2} |u_{s_i}(t_i) - \bar{u}_{s_i}(t_i)|$$

provided that $s_{i+1} = s_i$ and $t_{i+1} = t_i + 1$.

Thus, $|u_{s_n}(t_n) - \bar{u}_{s_n}(t_n)| \geq M^n |u_0 - \bar{u}_0| \geq M^n \hat{\varepsilon} > \mathcal{D}$, i.e. the points $u_{s_n}(t_n)$ and $\bar{u}_{s_n}(t_n)$ lie in different subintervals of I and therefore, $\omega^{t_n}_{s_n} \neq \bar{\omega}^{t_n}_{s_n}$. It follows from the definition of s_i and t_i that $| s_i - s_0 | \leq nr$ for any $1 \leq i \leq n$ and that $| t_n - t_0 | \leq n$. □

Lemma 4 *For any j and n in $\mathbb{Z}^+$ there exists $\delta > 0$ such that $D_q(\omega, \bar{\omega}) < \delta$ implies*

$$\omega_{\mathrm{s}}^t = \bar{\omega}_{\mathrm{s}}^t, \quad \| \mathrm{s} \| \leq nr + j, \quad 0 \leq t < n.$$

Proof. The condition $D(\omega, \bar{\omega}) < \delta$ implies that for all $\mathrm{s} \in \mathbb{Z}^d$, $q^{-\| \mathrm{s} \|} d(\omega_{\mathrm{s}}, \bar{\omega}_{\mathrm{s}}) < \delta$. The latter implies the existence of an integer S_δ such that if $\| \mathrm{s} \| \leq S_\delta$ then there exists $t_{\mathrm{s}} \geq 1$ such that

$$\omega_{\mathrm{s}}^t = \bar{\omega}_{\mathrm{s}}^t, \; 0 \leq t < t_{\mathrm{s}}.$$

The integer t_{s} is a non-increasing function of $\| \mathrm{s} \|$. Moreover, both S_δ and t_{s} go to infinity when δ goes to 0. Therefore, the window $\{(\mathrm{s}, t) \in \mathbb{Z}^{d+1} \; : \; \| \mathrm{s} \| \leq nr + j, \; 0 \leq t < n\}$ is contained in the window $\{(\mathrm{s}, t) \in \mathbb{Z}^{d+1} \; : \; \| \mathrm{s} \| \leq S_\delta, \; 0 \leq t \leq t_{\mathrm{s}}\}$ if δ is sufficiently small. □

We are ready to prove the continuity of the map Π in the product topology:

We have that $I_{\mathrm{s}}^t \subset I_{\mathrm{s}} \subset I = [a, b]$. For $\varepsilon > 0$ given, there exists $j^* \in \mathbb{Z}^+$ (which goes to infinity as ε goes to 0) such that

$$(b - a) \left(\sum_{i=j^*}^{\infty} \frac{(2i+1)^d - (2i-1)^d}{q^i} \right) < \frac{\varepsilon}{2}.$$

Choose $\hat{\varepsilon}$ such that

$$\hat{\varepsilon} \left(1 + \sum_{i=1}^{j^*-1} \frac{(2i+1)^d - (2i-1)^d}{q^i} \right) < \frac{\varepsilon}{2}.$$

By Corollary 1 there exists n such that for any configurations $\omega = \{\omega_{\mathrm{s}}^k\}_{\mathrm{s} \in \mathbb{Z}^d, k \in \mathbb{N}}$ and $\bar{\omega} = \{\bar{\omega}_{\mathrm{s}}^k\}_{\mathrm{s} \in \mathbb{Z}^d, k \in \mathbb{N}}$ in Σ satisfying $\omega_{\mathrm{s}}^t = \bar{\omega}_{\mathrm{s}}^t$ for $\| \mathrm{s} \| \leq nr + j^*$ and $0 \leq t < n$, we have

$$| \, (\Pi\omega)_{\mathrm{s}} - (\Pi\bar{\omega})_{\mathrm{s}} \, | < \hat{\varepsilon}, \quad \| \mathrm{s} \| \leq j^*.$$

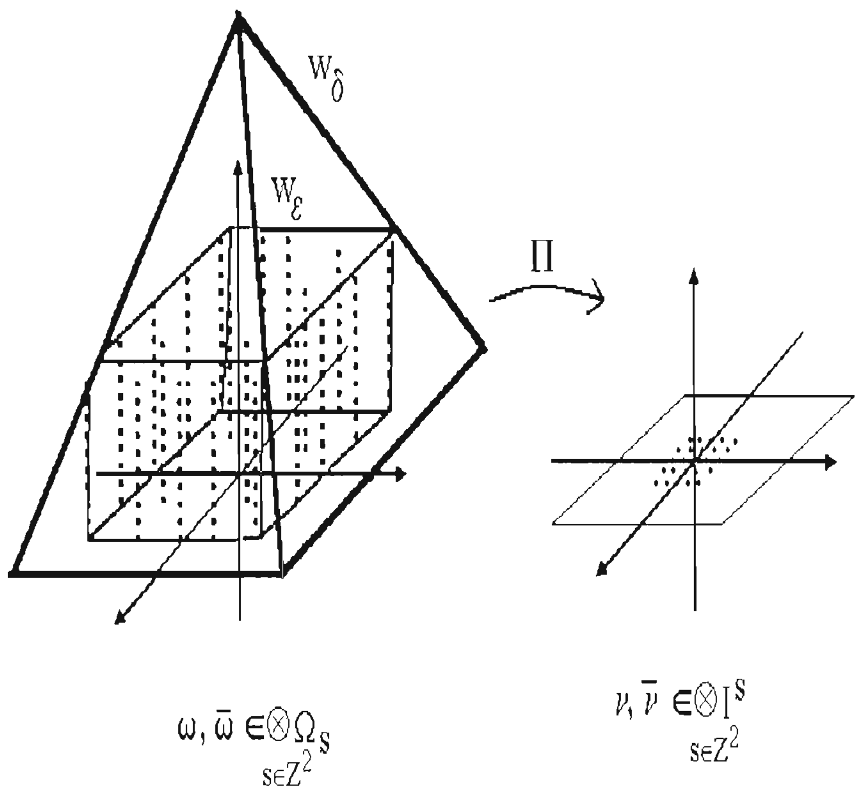

Figure 3.1: The continuity of the map Π.

By Lemma 4 there exists $\delta > 0$ such that $D_q(\omega, \bar{\omega}) < \delta$ implies

$$\omega_{\mathrm{s}}^t = \bar{\omega}_{\mathrm{s}}^t, \quad \| \mathrm{s} \| \leq nr + j^*, \quad 0 \leq t \leq n.$$

□

Chapter 4

Possible generalizations

We believe that the technique developed in the work (mainly lemmas 3, 4) will allow us to obtain some results in the following directions:

- Instead of consideration of repellers in 1–dimensional expanding local maps, one can try to consider individual p–dimensional systems $p > 1$, with hyperbolic invariant sets. For the beginning it is natural to consider individual maps with smale horseshoes.
- One can generalize results of [1] and [3] to study directional entropy and density of directional entropy for weakly coupled LDS in d–dimensional case, $d > 1$. For that, Theorem 1 provided in this work is the necessary first step. In this generalization, the case of local maps with regular local dynamics is of the main interest.
- The dynamics of networks of interactive active elements attracts an attention many specialists nowadays. The results of the work can be definitely generalized to the case where finitely many individual maps are interacting with each other by a rule defined through a graph of interactions. Moreover, some specific problems for networks of coupled elements can be rigorously formulated and solved.

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