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Boolean algebras in the study of Bell inequalities

Tesis para obtener el grado de

Maestro en Ciencias (Física)

Presenta:

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24 de Agosto del 2012

Abstract

This thesis shows that the following statements are equivalent:

1. There is a hidden-variable model based on Local Realism for the experiment.

2. The experiment supports a local-realistic boolean probability algebra.

3. Correlation Function for the experiment satisfies Bell-CHSH inequalities.

The equivalence is obtained by algebraic methods. In particular, this work gives a merely algebraic proof of Bell-CHSH inequality.

A la memoria de mi papá, Fernando Méndez Romo.

Para mi mamá, Rosita, con toda mi gratitud, admiración y amor.

En la vastedad del espacio y en la inmesidad del tiempo es para mi un placer haber compartido un lugar y una época con ellos.

Contents

Introduction

From the publication of Bell's Theorem in 1965 [1], which states that no theory involving *locality* and *realism* can reproduce the same statistical predictions of Quantum Mechanics for the Bohm-EPR Experiment [2], [3], criteria to test the local-realistic character of this system have been sought.

The first attempt to classically explain this experiment were the hidden-variable physical models, which emphasized the production process. Bell proposed a Correlation Function for these models [1], which introduced a probability density function and a phase space that fitted a boolean probability algebra.

Years later, Clause, Horne, Shimony and Holt (CHSH) showed that this Correlation Function satisfies the well-known Bell-CHSH inequality [4], which is a necessary [4] and sufficient [5] criterion to test the local-realistic character of the aforementioned experiment.

Then, the equivalence between the boolean probability algebra supported by the experiment and Bell-CHSH inequality passed through the existence of hidden-variables, which made them the essential part of the problem.

The aim of this thesis is to find the direct equivalence between the local-realistic boolean probability algebra supported by the experiment and Bell-CHSH inequality. This is achieved by means of a merely algebraic proof of this inequality. In addition, it is shown that regardless of the production process, as long as it is real and local, the probabilities involved by any hidden-variable physical model correspond to those ones of a local-realistic boolean probability algebra.

In this way, the equivalence among the hidden-variable models, the boolean probability algebra supported by the experiment and Bell-CHSH inequality is obtained by using algebraic methods, thus showing that the essential part of the problem lies on the probability algebra supported by the experiment and not on the hidden-variables nor on the production process.

This thesis is divided in four chapters. Chapter 1 comprise the Bohm-EPR experiment, the hidden-variable physical models, Bell´s Theorem and the analytic proof of Bell-CHSH inequality; Chapter 2 contains the construction of the local-realistic boolean probability algebra for the experiment; Chapter 3 is the merely algebraic proof of Bell-CHSH inequality and finally Chapter 4 shows the equivalence between the hidden-variable physical models and the local-realistic boolean probability algebras supported by the experiment.

Chapter 1

The Bohm-EPR experiment

The aim of this chapter is to explain the Bohm-EPR experiment [2], [3] and to introduce the local-realistic hidden-variable physical models, and from these ones to obtain Bell-CHSH inequality [4], which is a necessary [4] and sufficient [5] criterion to test the local-realistic character of this experiment.

1.1 Description of the Bohm-EPR experiment [2], [3], [6]

Consider an ensemble of pairs of *spin-*$\frac{1}{2}$ *particles* in singlet spin state moving freely in opposite direction from the source that produced them. Measurements of the particle 1 are made by the observer *Alice* and measurements of the particle 2 are made by the observer *Bob*. They both have a Stern-Gerlach (SG) apparatus and a detector (see Figure 1.1). Together, observer, SG apparatus and detector, form a subsystem, or *site*, which will be denoted as A for Alice's system and B for Bob's system. Hence, we will say that measurements of particle 1 are made at site A and measurements of particle 2 are made at site B.

SG apparatus are oriented in directions $\vec{a}$ and $\vec{b}$ for sites A and B, respectively. The detectors used for counting particles have, in turn, two outcome channels, labelled as $+1$ and -1 (see Figure 1.1).

Let us suppose that the results for A and B depend on both SG apparatus choices $\vec{a}$ and $\vec{b}$. Then, they will be represented by $A(\vec{a},\vec{b})$ and $B(\vec{a},\vec{b})$, each one equals $+1$ or -1 depending on whether the first or second channel is selected, i.e.:

$$A(\vec{a},\vec{b}) = \begin{cases} +1 & \text{if partcile 1 hits channel } +1. \\ -1 & \text{if particle 1 hits channel -1.} \end{cases} \tag{1.1.1a}$$

$$B(\vec{a},\vec{b}) = \begin{cases} +1 & \text{if partcile 2 hits channel } +1. \\ -1 & \text{if particle 2 hits channel -1.} \end{cases} \tag{1.1.1b}$$

Thus, in a single run we can obtain one of the following four results (simple events):

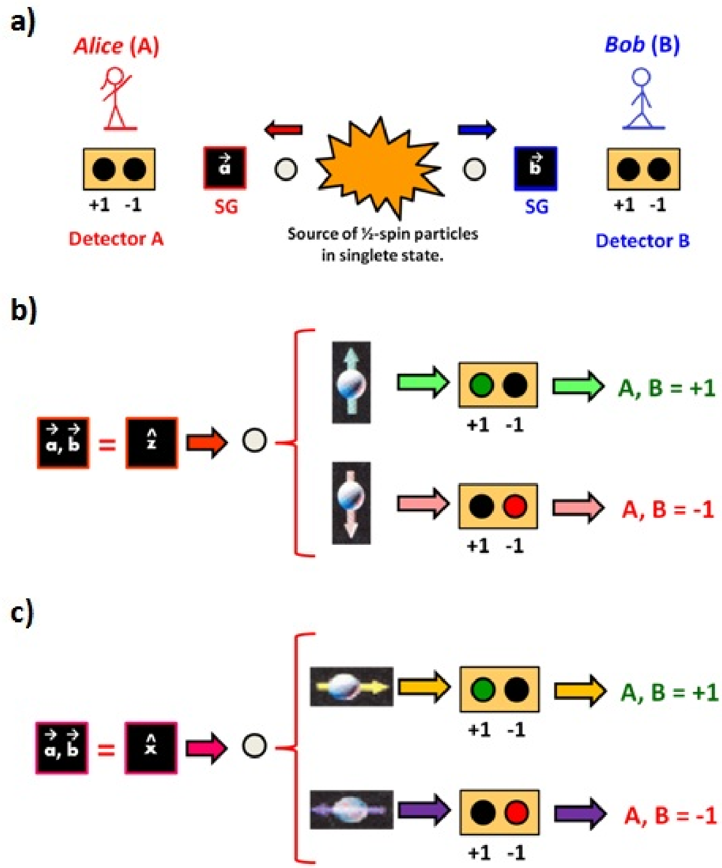

Figure 1.1: **Description of the experiment.** ***a)*** A pair of spin-$\frac{1}{2}$ particles in singlet spin state is moving freely in opposite direction from the source that produced them. Each one will travel towards a Stern-Gerlach (SG) apparatus, which is oriented in the direction $\vec{a}$ for site A and in the direction $\vec{b}$ for site B. Finally, measurements are made by detectors with two outcomes. Depending on the orientation of the SG apparatus, one of the two outcomes will be activated: $+1$ (green light in the picture) or -1 (red light in the picture), assigning such value to the measurement of A or B, as required. ***b)*** We illustrate the particular case when the chosen direction for any SG apparatus is $\hat{z}$ (i.e., $\vec{a}$ or $\vec{b} = \hat{z}$). ***c)*** Same case, but now the chosen orientation is $\hat{x}$ (i.e., $\vec{a}$ or $\vec{b} = \hat{x}$).

$$(A(\vec{a},\vec{b}) = +1,\, B(\vec{a},\vec{b}) = +1)\ , \tag{1.1.2a}$$
$$(A(\vec{a},\vec{b}) = +1,\, B(\vec{a},\vec{b}) = -1)\ , \tag{1.1.2b}$$
$$(A(\vec{a},\vec{b}) = -1,\, B(\vec{a},\vec{b}) = +1)\ , \tag{1.1.2c}$$
$$(A(\vec{a},\vec{b}) = -1,\, B(\vec{a},\vec{b}) = -1)\ . \tag{1.1.2d}$$

Furthermore, for a *particular measurement*, we will say that a pair of particles has a positive correlation if they both choose the same channel, and a negative correlation if they both choose a different channel. In other words:

$$A(\vec{a},\vec{b})B(\vec{a},\vec{b}) = \begin{cases} +1 & \text{for a positive correlation.} \\ -1 & \text{for a negative correlation.} \end{cases} \tag{1.1.3}$$

Due to the fact that the former product only has two possible values (+1 and -1), we are rather interested in measuring *compound events*, in particular, those with one of the following characteristics:

1. $A(\vec{a},\vec{b}) = B(\vec{a},\vec{b})$, or
2. $A(\vec{a},\vec{b}) = -B(\vec{a},\vec{b})$.

The compound events with these characteristics are the following ones:

$$\varepsilon_{ab} \equiv (A(\vec{a},\vec{b}) = +1, B(\vec{a},\vec{b}) = +1) \vee (A(\vec{a},\vec{b}) = -1, B(\vec{a},\vec{b}) = -1) \ , \tag{1.1.4a}$$

$$\delta_{ab} \equiv (A(\vec{a},\vec{b}) = +1, B(\vec{a},\vec{b}) = -1) \vee (A(\vec{a},\vec{b}) = -1, B(\vec{a},\vec{b}) = +1) \ , \tag{1.1.4b}$$

where $\vee$ is the usual OR connector.

In this way:

$$A(\vec{a},\vec{b})B(\vec{a},\vec{b}) = \begin{cases} +1 & \text{for } \varepsilon_{ab}. \\ -1 & \text{for } \delta_{ab}. \end{cases} \tag{1.1.5}$$

Definition 1. ***Experimental correlation*** *(M_{ab}): is the statistical average of the product $A(\vec{a},\vec{b})B(\vec{a},\vec{b})$ after repeating the experiment many times, i.e.:*

$$M_{ab} \equiv \overline{A(\vec{a},\vec{b})\ B(\vec{a},\vec{b})} \tag{1.1.6}$$

Furthermore, as we have that $A(\vec{a},\vec{b})B(\vec{a},\vec{b}) = \{+1,-1\}$, M_{ab} (1.1.6) has a numerical value between -1 and $+1$, i.e., $-1 \leqslant M_{ab} \leqslant +1$, where $+1$ is gotten for a totally correlated system, while -1 is obtained for a totally anticorrelated system.

It is important to say that M_{ab} (1.1.6) is a value that the experimentalist measures in the lab. This value, once determined, completely characterizes the correlations system.

According to (1.1.6), we must count how many times each simple event (1.1.2), or rather, each compound event (1.1.4), occurs when we perform the experiment many times, i.e., we must find the frequencies for each of these events. Then, the frequencies for ε_{ab} and δ_{ab} are given by:

$$f_{\varepsilon_{ab}} \equiv \frac{number\ of\ times\ that\ \varepsilon_{ab}\ is\ obtained}{number\ of\ total\ runs} \tag{1.1.7a}$$

$$f_{\delta_{ab}} \equiv \frac{number\ of\ times\ that\ \delta_{ab}\ is\ obtained}{number\ of\ total\ runs} \tag{1.1.7b}$$

Thereby, we have the following relation between the experimental correlation M_{ab} (1.1.6) and the frequencies $f_{\varepsilon_{ab}}$ and $f_{\delta_{ab}}$ (1.1.7):

$$M_{ab} = f_{\varepsilon_{ab}} - f_{\delta_{ab}} \tag{1.1.8}$$

Then, as it is well known, we can associate these frequencies with probabilities in such a way that we can get a probabilistic model capable to predict the experimental value M_{ab}. Thus, the task is now finding such a probability distribution.

1.2 Classical description of the Bohm-EPR experiment

1.2.1 Realism and Locality Hypotheses

The first probabilistic model we present is that one suggested by EPR [2]. They said that the probability distribution for this problem should be given by any model based on two important hypotheses: ***Realism*** and ***Locality***.

Hypothesis 1. ***Realism.*** *Each time a particular run is performed, the values for* $A(\vec{a},\vec{b})$ *and* $B(\vec{a},\vec{b})$ *that correspond to each of the directions* $\vec{a}$ *and* $\vec{b}$ *are determined, although they are not measured. In other words, for every experimental run the assignation* $\{\vec{a},\vec{b}\} \longmapsto \{A(\vec{a},\vec{b}),B(\vec{a},\vec{b})\}$ *is done for all the values of* $\vec{a}$ *and* $\vec{b}$*.*

A way (and for many years the only one) to establish the realism into the problem is through a ***deterministic description***, known as ***hidden-variable model***, which says that a complete specification of the system is effected by means of a parameter (or set of parameters) λ, named ***hidden-variable***. This λ is information (emphatically not quantum mechanics) carried by and localized within each particle, and it was originated when the particles constituting one pair were in contact and communication regarding this information. If we were able to determine λ, we could know the values of $A(\vec{a},\vec{b})$ and $B(\vec{a},\vec{b})$ for any choice of $\vec{a}$ and $\vec{b}$ for particular measurements, avoiding in this way the statistical behaviour of Quantum Mechanics. In this model, the map is described as $\{\vec{a},\vec{b}\} \overset{\lambda}{\longmapsto} \{A(\lambda,\vec{a},\vec{b}),B(\lambda,\vec{a},\vec{b})\}$; in other words, the assignation of values for A and B from $\vec{a}$ and $\vec{b}$ is carried out by λ.

The other hypothesis is ***Locality***.

Hypothesis 2. ***Locality.*** *In the experimental setup of the Bohm-EPR experiment, sites* A *and* B *can arbitrarily be far away from each other, in such a way that there is no communication between them. Then, the particular orientation of a SG magnet should not influence the result obtained in the opposite site. Then, we have that* $A(\vec{a},\vec{b}) \equiv A(\vec{a})$ *and* $B(\vec{a},\vec{b}) \equiv B(\vec{b})$*.*

In this chapter we will develop these hypotheses in the same way Bell [1], CHSH [4] and Fine [7] did. Later on in Chapter 2 we will present a new model based on probabilities defined over a boolean algebra of events, showing in this manner that λ is not the essential part of the problem.

1.2.2 Hidden-variable models for the Bohm-EPR experiment

In 1965, Bell [1] found a general way to describe correlations in the Bohm-EPR experiment for *any* local-realistic hidden-variable physical model. Such a way is known as *Correlation Function*.

Definition 2. *The **Correlation Function** Q_{ab} that is consistent with Hypothesis 1 (Realism) and 2 (Localty) has the following form:*

$$Q(\vec{a},\vec{b}) = \int_\Gamma A(\vec{a},\lambda)B(\vec{b},\lambda)\rho(\lambda)\ d\lambda\ , \tag{1.2.1}$$

where ρ is the probability density function of λ and Γ is the λ's phase-space.

As we can see, ρ is independent of $\vec{a}$ and $\vec{b}$ since the pair of particles is emitted by a source in a manner physically independent from them.

From now on, it is convenient to use a new nomenclature:

$$A_a \equiv A(\vec{a},\lambda)\ , \tag{1.2.2a}$$

$$B_b \equiv B(\vec{b},\lambda)\ , \tag{1.2.2b}$$

$$Q_{ab} \equiv Q(\vec{a},\vec{b})\ , \tag{1.2.2c}$$

$$\int_\Gamma A_a\ B_b\ d\rho \equiv \int_\Gamma A(\vec{a},\lambda)B(\vec{b},\lambda)\rho(\lambda)d\lambda\ . \tag{1.2.2d}$$

In this notation, we have that the subscript a refers to the orientation of the SG apparatus at site A, while the subscript b refers to the orientation of the SG apparatus at site B.

The Correlation Function Q_{ab} (1.2.1) is the prediction for the product of A_a and B_b, and it *only* depends on the probability density function ρ, *not on the quantum mechanical state of the pair.*

As we can see, A_a and B_b now depend on the parameter λ because we suppose it carries the information that relates both particles and determines the specific configuration of the system for individual measurements. In this way, the system would be predetermined by the common past of the particles. On the other hand, locality is present because A_a does not depend on $\vec{b}$, nor B_b on $\vec{a}$, since the two selections may occur at an arbitrarily great distance from each other; locality also makes A_a and B_b to be independent functions, i.e., we are refusing any kind of dependence $A_a(B_b)$ or $B_b(A_a)$. So, the Correlation Function Q_{ab} defined in the manner (1.2.1) includes *every kind of models involving locality and determinism.* A different model, a different ρ.

In a complete physical theory, hidden-variables would have dynamical significance and be subjected to laws of motion; our λ can then be thought of as initial values of these variables at some suitable instant and ρ would be an invariant probability density.

The following 3-parameter inequality, which was used by Bell in his original article [1], is an inherent property from the definition of the Correlation Function Q_{ab} (1.2.1), which means it is satisfied by *any* hidden-variable model consistent with *Hypothesis 1* and *2*, and it is totally independent of Quantum Mechanics.

Lemma 1. *For every choice of vectors $\vec{a},\vec{b}$ and $\vec{c}$, and for all ρ, the Correlation Function Q satisfies the following 3-parameter inequality:*

$$\Big|Q_{ab} - Q_{ac}\Big| \leqslant 2 + Q_{bc} + Q_{bb}\ , \tag{1.2.3}$$

Proof. From *Definition 2*, we have:

$$Q_{ab} = \int_{\Gamma} A_a \, B_b \, d\rho \; , \tag{1.2.4}$$

where $A_a, B_b = \{+1, -1\}$.

Then:

$$\begin{aligned} Q_{ab} - Q_{ac} &= \int_{\Gamma} A_a \, B_b \, d\rho \; - \; \int_{\Gamma} A_a \, B_c \, d\rho \\ &= \int_{\Gamma} A_a \, B_b \, \big(1 + A_b \, B_c\big) \, d\rho \; - \; \int_{\Gamma} A_a \, B_c \, \big(1 + A_b \, B_b\big) \, d\rho \; , \end{aligned} \tag{1.2.5}$$

where $\vec{b}$ and $\vec{c}$ are the orientations of the SG magnet at site B for different experimental runs.

Making use of the known inequality $\Big| \int f(x) \, dx \Big| \leqslant \int \Big| f(x) \Big| \, dx$ and of $\Big| A_a \Big| = 1$, we get:

$$\Big| Q_{ab} - Q_{ac} \Big| \leqslant \int_{\Gamma} \Big| B_b \, \big(1 + A_b \, B_c\big) \; - \; B_c \, \big(1 + A_b \, B_b\big) \Big| \, d\rho \; . \tag{1.2.6}$$

Knowing that $\big| x - y \big| \leqslant \big| x \big| + \big| y \big|$, where $x, \, y \in \mathbb{R}$, we have:

$$\begin{aligned} \Big| Q_{ab} - Q_{ac} \Big| &\leqslant \int_{\Gamma} \Big| B_b \, (1 + A_b \, B_c) \Big| \, d\rho \; + \; \int_{\Gamma} \Big| B_c \, (1 + A_b \, B_b) \Big| \, d\rho \\ &= 2 \; + \; \int_{\Gamma} A_b \, B_c \, d\rho \; + \; \int_{\Gamma} A_b \, B_b \, d\rho \; , \end{aligned} \tag{1.2.7}$$

where we have used that $\big| B_b \big| = \big| B_c \big| = 1$, and that the quantities in parentheses are never negative.

Finally, using the definition of the Correlation Function Q_{ab} (1.2.1), we obtain:

$$\Big| Q_{ab} - Q_{ac} \Big| \leqslant 2 + Q_{bc} + Q_{bb} \; , \tag{1.2.8}$$

which is the expected result.

□

1.3 Quantum mechanical description of the Bohm-EPR experiment

There is another model to explain the Bohm-EPR experiment based on quantum mechanical principles.

In first place, this system consists on spin-$\frac{1}{2}$ particles in singlet state, i.e., particles which vector state is given by:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_A |-\rangle_B \; - \; |-\rangle_A |+\rangle_B \right) \; , \tag{1.3.1}$$

or, in matrix representation:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B \; - \; \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B \right] \; , \tag{1.3.2}$$

where the subscript indicates the particle we are referring to.

In the quantum case, instead of $A(\vec{a}, \vec{b})$ and $B(\vec{a}, \vec{b})$, we are interested in measuring the c-component of the orientation vector $\vec{\sigma_C}$, where $\vec{c} = \vec{a}, \vec{b}$ refers to the direction of the SG-magnet at site C=A,B, respectively.

From this point of view, the SG apparatus selects a direction to measure on the components of the orientation vectors $\vec{\sigma_A}$ and $\vec{\sigma_B}$ for particles at sites A and B, respectively. These orientation vectors have as components the Pauli Matrices. Let $\vec{c}$ be a unit vector that will be $\vec{a}$ for site A and $\vec{b}$ for site B. Then, the c-component of these orientation vectors $\vec{\sigma}$ is given by:

$$\vec{\sigma} \cdot \vec{c} = \begin{pmatrix} c_z & c_x - i\, c_y \\ c_x + i\, c_y & -c_z \end{pmatrix} \; , \tag{1.3.3}$$

or:

$$\langle + | \vec{\sigma} \cdot \vec{c} | + \rangle = c_z \; , \quad \langle + | \vec{\sigma} \cdot \vec{c} | - \rangle = c_x - i\, c_y \; ,$$

$$\langle - | \vec{\sigma} \cdot \vec{c} | + \rangle = c_x + i\, c_y \; , \quad \langle - | \vec{\sigma} \cdot \vec{c} | - \rangle = -c_z \; . \tag{1.3.4}$$

In this case, $\vec{a}$ and $\vec{b}$ are the directions of the components we are measuring in particles at sites A and B, respectively.

Furthermore, in this quantum model we say that the outcome channels of the detectors are selected according to the eigenvectors chosen by the SG apparatus.

Theorem 1. *The Quantum Correlation Function R_{ab} is given by:*

$$R_{ab} \equiv \langle \vec{\sigma_A} \cdot \vec{a} \quad \vec{\sigma_B} \cdot \vec{b} \rangle = - \vec{a} \cdot \vec{b} \, , \tag{1.3.5}$$

where the quantum expectation value is taken with respect to the singlet state (1.3.1).

Proof. Taking the expectation value $\langle \vec{\sigma_A} \cdot \vec{a} \quad \vec{\sigma_B} \cdot \vec{b} \rangle$ with respect to the singlet state (1.3.1) and using (1.3.4) with $\vec{c} = \vec{a}, \vec{b}$, we obtain:

$$\langle \psi | \vec{\sigma_A} \cdot \vec{a} \quad \vec{\sigma_B} \cdot \vec{b} | \psi \rangle = -a_z b_z - a_y b_y - a_x b_x = - \vec{a} \cdot \vec{b} \; , \tag{1.3.6}$$

which is the expected result. □

1.4 Bell's Theorem

In the previous sections, we have described two very different models to foretell the measurements of the same experiment. We now compare such predictions to see if they are compatible for all $\vec{a}$ and $\vec{b}$ for any ρ, independently of the value for M_{ab} obtained in the experiment. If they do, we then have that Quantum Mechanics can be "explained" in terms of *Hypothesis 1* and *2* for this particular problem.

In 1965 Bell [1] compared both models and found a specific configuration of parameters $\vec{a}$ and $\vec{b}$ for which is impossible for any ρ that both models predict the same results. In this way, he showed that, for this particular problem, Quantum Mechanics cannot be "explained" in terms of any model involving locality and realism. In other words, he showed that R_{ab} cannot be arbitrarily closely approximated by Q_{ab}.

Theorem 2. ***Bell's Theorem.*** *There are vectors $\vec{a}$ and $\vec{b}$ such that the quantum mechanical expectation value R_{ab} cannot be arbitrarily closely approximated for any ρ by the local-realistic average Q_{ab}.*

Proof. Instead of Q_{ab} and R_{ab}, let us consider the averages $\overline{Q_{ab}}$ and $\overline{R_{ab}}$, where the bar denotes *independent averaging* of $Q_{a'b'}$ and $R_{a'b'}$ over vectors $\vec{a'}$ and $\vec{b'}$ within specified small angles of $\vec{a}$ and $\vec{b}$, which means we average over cones with center in $\vec{a}$ and $\vec{b}$ (see Figure 1.2).

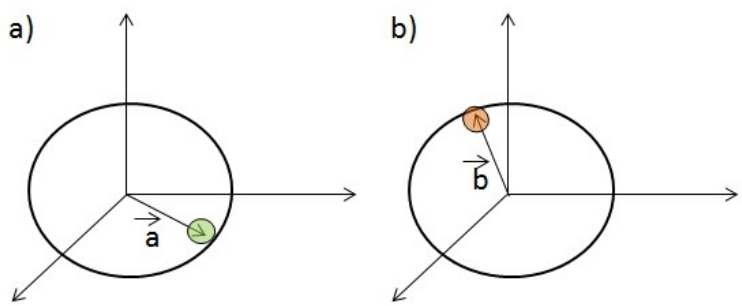

Figure 1.2: **a)** Party A takes the average over a cone centred on the vector $\vec{a}$. **b)** Party B takes the average over a cone centred on the vector $\vec{b}$.

Then, for all $\vec{a}$ and $\vec{b}$, and $\delta > 0$, there exist cones coaxial to $\vec{a}$ and $\vec{b}$ such that:

$$\left| \overline{R_{ab}} - R_{ab} \right| \leqslant \delta \; . \tag{1.4.1}$$

This is a technical step to analyze the dispersion of the values, i.e., how width the cone over which we average is. Averages are taken over sets of positive size to avoid individual measurements and work with mean values (a individual measurement has size or probability zero).

The proof proceeds by contradiction.

Suppose that for every choice of vectors $\vec{a}$ and $\vec{b}$, and for every $\epsilon > 0$, there is a ρ and a $\delta > 0$ for which:

$$\left|\overline{Q_{ab}} - \overline{R_{ab}}\right| \leqslant \epsilon \ . \tag{1.4.2}$$

This expression is to compare the predictions of both models: we want to see how far the *mean values* predicted by each model are. We will show that there are vectors $\vec{a}$ and $\vec{b}$ for which (1.4.2) is not true for every $\epsilon > 0$.

The difference between expressions (1.4.1) and (1.4.2) is that, while the first one assumes the existence of cones coaxial to $\vec{a}$ and $\vec{b}$ for every δ, the second one assumes the existence of δ and ρ for every $\epsilon > 0$.

Adding (1.4.1) and (1.4.2), and using the triangle inequality we get:

$$\left|\overline{Q_{ab}} - R_{ab}\right| \leqslant \epsilon + \delta \ . \tag{1.4.3}$$

It is specially important the case $\vec{a} = \vec{b}$. So, making $\vec{a} = \vec{b}$ in (1.4.3), we obtain:

$$\overline{Q_{bb}} + 1 \leqslant \epsilon + \delta \ . \tag{1.4.4}$$

Coming back to (1.4.3), let us express it in a different form:

$$\begin{gathered} -(\epsilon + \delta) \leqslant \overline{Q_{ab}} - R_{ab} \leqslant (\epsilon + \delta) \\ -(\epsilon + \delta) + R_{ab} \leqslant \overline{Q_{ab}} \leqslant (\epsilon + \delta) + R_{ab} \ . \end{gathered} \tag{1.4.5}$$

In a similar way:

$$-(\epsilon + \delta) + R_{ac} \leqslant \overline{Q_{ac}} \leqslant (\epsilon + \delta) + R_{ac} \ . \tag{1.4.6}$$

Multiplying (1.4.6) by (-1):

$$-(\epsilon + \delta) - R_{ac} \leqslant -\overline{Q_{ac}} \leqslant (\epsilon + \delta) - R_{ac} \ . \tag{1.4.7}$$

Adding (1.4.5) and (1.4.7):

$$\begin{gathered} -2(\epsilon + \delta) + (R_{ab} - R_{ac}) \leqslant \overline{Q_{ab}} - \overline{Q_{ac}} \leqslant 2(\epsilon + \delta) + (R_{ab} - R_{ac}) \\ \left|\left(R_{ab} - R_{ac}\right) - \left(\overline{Q_{ab}} - \overline{Q_{ac}}\right)\right| \leqslant 2(\epsilon + \delta) \ . \end{gathered} \tag{1.4.8}$$

Using the relation $\left|x\right| - \left|y\right| \leqslant \left|x - y\right|$, where x, $y \in \mathbb{R}$, we get:

$$\left|R_{ab} - R_{ac}\right| - \left|\overline{Q_{ab}} - \overline{Q_{ac}}\right| \leqslant 2(\epsilon + \delta) \ . \tag{1.4.9}$$

From here:

$$\left|R_{ab} - R_{ac}\right| - 2(\epsilon + \delta) \leqslant \left|\overline{Q_{ab}} - \overline{Q_{ac}}\right| . \tag{1.4.10}$$

On the other hand, from (1.4.5), making $a \longrightarrow b$ and $b \longrightarrow c$, we have:

$$\overline{Q_{bc}} \leqslant (\epsilon + \delta) + R_{bc} . \tag{1.4.11}$$

Furthermore, from *Lemma 1* (1.2.3), it turns out that:

$$\left|\overline{Q_{ab}} - \overline{Q_{ac}}\right| \leqslant 2 + \overline{Q_{bc}} + \overline{Q_{bb}} , \tag{1.4.12}$$

which is an inherent property from the definition of the Correlation Function Q_{ab} (1.2.1).

Substituting (1.4.4) in this expression we get:

$$\left|\overline{Q_{ab}} - \overline{Q_{ac}}\right| \leqslant 1 + \overline{Q_{bc}} + \epsilon + \delta . \tag{1.4.13}$$

Replacing (1.4.10) and (1.4.11) in this equation, we obtain.

$$\left|R_{ab} - R_{ac}\right| - R_{bc} - 1 \leqslant 4(\epsilon + \delta) . \tag{1.4.14}$$

Finally, making $R_{ab} = -\vec{a} \cdot \vec{b}$ (1.3.5), we get:

$$\left|\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}\right| + \vec{b} \cdot \vec{c} - 1 \leqslant 4(\epsilon + \delta) . \tag{1.4.15}$$

For example, if $\vec{a} \cdot \vec{c} = 0$, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \frac{1}{\sqrt{2}}$, we have:

$$\sqrt{2} - 1 \leqslant 4(\epsilon + \delta) , \tag{1.4.16}$$

which is independent of ρ.

Therefore, for arbitrarily small finite δ, ϵ cannot arbitrarily be small. It means that (1.4.2) is not true for values of ϵ such that:

$$\epsilon < \frac{\sqrt{2} - 1}{4} , \tag{1.4.17}$$

whatever the local-realistic state ρ is. □

In this way, Bell found a specific experimental setup such that the predicted results from any local-realistic hidden-variable physical model differ at least by $\frac{\sqrt{2}-1}{4}$ from those predicted by Quantum Mechanics.

Thus, the quantum mechanical expectation value R_{ab} cannot be represented either accurately or arbitrarily closely by the local-realistic average Q_{ab}.

1.5 Testing hidden-variables

So far, we have just showed that the results predicted by the local-realistic Correlation Function Q_{ab} differ for some $\vec{a}$ and $\vec{b}$ from the quantum mechanical expectation value R_{ab}, but we have said nothing about which model is correct.

In 1969, Clauser, et al. (CHSH) [4] found the following *necessary* [4] and *sufficient [5] condition* for *any* local-realistic hidden-variable model for this experiment, which also is a test to experimentally verify if this particular problem can be explained in terms of realism and locality.

For simplicity, from now on we will make reference to the vectors that represent the orientation of the SG magnets as parameters of the experiment.

Theorem 3. ***Bell-CHSH inequality.*** *For any choice of parameters a, b, c and d, and for all ρ, the Correlation Function Q satisfies the following 4-parameter inequality:*

$$\Big|Q_{ab} - Q_{ac}\Big| + Q_{db} + Q_{dc} \leqslant 2 \; . \tag{1.5.1}$$

Proof. From *Definition 2* (1.2.1)

$$\Big|Q_{ab} - Q_{ac}\Big| = \Big| \int_{\Gamma} (A_a \; B_b - A_a \; B_c) \; d\rho \; \Big| \; . \tag{1.5.2}$$

Due to the known inequality $\Big| \int f(x)dx\Big| \leqslant \int \Big|f(x)\Big|dx$, we have that:

$$\begin{aligned} \Big|Q_{ab} - Q_{ac}\Big| &\leqslant \int_{\Gamma} \Big|A_a \; B_b - A_a \; B_c\Big| \; d\rho \\ &\leqslant \int_{\Gamma} \Big|A_a \; B_b\Big|\Big(1 - B_b \; B_c\Big) \; d\rho \\ &\leqslant \int_{\Gamma} \Big(1 - B_b \; B_c\Big) \; d\rho \\ &\leqslant 1 \; - \int_{\Gamma} B_b \; B_c \; d\rho \; , \end{aligned} \tag{1.5.3}$$

where we have used the fact that ${B_b}^2 = 1$ and that $\Big|A_a \; B_b\Big| = 1$.

Thus:

$$\Big|Q_{ab} - Q_{ac}\Big| \leqslant 1 - \int_\Gamma B_b\, B_c\, d\rho\ . \tag{1.5.4}$$

Notice that the left hand side of (1.5.4) depends on a, while the right hand side does not.

On the other hand, let us now divide Γ into two regions Γ_+ and Γ_- such that:

$$\Gamma_\pm \equiv \big\{\lambda \big| A_d = \pm B_b\big\}\ . \tag{1.5.5}$$

As we can see, Γ_+ and Γ_- are disjoint subsets of the phase space, with the property that $\Gamma_+ \cup \Gamma_- = \Gamma$.

Then, using (1.5.5) and the properties of the probability distribution functions, we get:

$$\begin{aligned}
\int_\Gamma B_b\, B_c\, d\rho &= \int_{\Gamma_+} B_b\, B_c\, d\rho\ + \int_{\Gamma_-} B_b\, B_c\, d\rho \\
&= \int_{\Gamma_+} A_d\, B_c\, d\rho\ - \int_{\Gamma_-} A_d\, B_c\, d\rho \\
&= \int_{\Gamma} A_d\, B_c\, d\rho\ - 2\int_{\Gamma_-} A_d\, B_c\, d\rho\ .
\end{aligned} \tag{1.5.6}$$

The first term of the right hand side of the last equation corresponds to Q_{dc}, while for the second one we have that:

$$\int_{\Gamma_-} A_d\, B_c\, d\rho \leqslant \int_{\Gamma_-} \big|A_d\, B_c\big|\, d\rho \leqslant \int_{\Gamma_-} d\rho\ , \tag{1.5.7}$$

where we have used again the relation $\Big|\int f(x)dx\Big| \leqslant \int \Big|f(x)\Big| dx$ and that $\Big|A_d\, B_c\Big| = 1$.

In this way, we get:

$$\int_\Gamma B_b\, B_c\, d\rho\ \geqslant\ Q_{dc}\ -\ 2\int_{\Gamma_-} d\rho\ . \tag{1.5.8}$$

On the other hand, from the properties of the probability distribution functions, we have that:

$$\int_\Gamma d\rho\ = \int_{\Gamma_+} d\rho\ + \int_{\Gamma_-} d\rho\ =\ 1\ . \tag{1.5.9}$$

From here:

$$\int_{\Gamma_+} d\rho\ = 1\ - \int_{\Gamma_-} d\rho\ . \tag{1.5.10}$$

Moreover, using again (1.5.5), we obtain:

$$\begin{aligned} Q_{db} &= \int_{\Gamma} A_d \, B_b \, d\rho \\ &= \int_{\Gamma_+} A_d \, B_b d\rho \, + \int_{\Gamma_-} A_d \, B_b \, d\rho \\ &= \int_{\Gamma_+} B_b \, B_b \, d\rho \, - \int_{\Gamma_-} B_b \, B_b \, d\rho \\ &= \int_{\Gamma_+} d\rho \, - \int_{\Gamma_-} d\rho \, . \end{aligned} \tag{1.5.11}$$

Substituting (1.5.10) in the last equation, we get:

$$Q_{db} = 1 \, - \, 2 \int_{\Gamma_-} d\rho \, . \tag{1.5.12}$$

Furthermore, we can express Q_{db} in a different way:

$$Q_{db} = -1 + \gamma \, , \tag{1.5.13}$$

where γ is a parameter such that $0 \leqslant \gamma \leqslant 2$ ($\gamma = 0$ for a totally anticorrelated system and $\gamma = 2$ for a totally correlated system).

Equating (1.5.12) and (1.5.13) we obtain:

$$\int_{\Gamma_-} d\rho = 1 - \frac{\gamma}{2} \, . \tag{1.5.14}$$

Replacing this expression in (1.5.8), we get:

$$\int_{\Gamma} B_b \, B_c \, d\rho \, \geqslant \, Q_{dc} + \gamma - 2 \, . \tag{1.5.15}$$

Substituting γ from (1.5.13) in this expression, we have that:

$$\int_{\Gamma} B_b \, B_c \, d\rho \, \geqslant \, Q_{dc} + Q_{db} - 1 \, . \tag{1.5.16}$$

Finally, replacing this equation in (1.5.4), we obtain:

$$\left| Q_{ab} - Q_{ac} \right| + Q_{db} + Q_{dc} \leqslant 2 \tag{1.5.17}$$

which is the expected result. □

We developed this inequality directly from the definition of the Correlation Function Q_{ab} (1.2.1). Thus, if it is verified in the experiment, we will have shown that there is *any* model involving locality and realism for this particular problem; otherwise, we will have shown that such a model does not exist.

This step is a great achievement because the Correlation Function Q_{ab} includes *any* hidden-variable physical model involving locality and realism.

A fundamental difference between this last inequality (1.5.1) and that one founded by Bell (1.2.3) is that this one is a 4-parameter inequality, while the other one only depends on 3.

In this chapter Bell-CHSH inequality was gotten by assuming the existence of hidden-variables. Later on in Chapter 3 we will give a merely algebraic proof of this inequality, showing that the essential part of the problem lies on the probability algebra of the experiment and not on the hidden-variables.

Chapter 2

Boolean algebras and local realism

The aim of this chapter is to show that Bohm-EPR experiment supports a local-realistic boolean probability algebra for the first non-trivial case (2 measurement parameters per site).

In this new approach, A_a and B_b refer only to $A(\vec{a})$ and $B(\vec{b})$ and not to $A(\vec{a}, \lambda)$ and $B(\vec{b}, \lambda)$ any more because they do not make reference to any hidden variable.

In addition, from now on we will use for simplicity the following nomenclature for the simple events from (1.1.2):

$$(+1, +1)_{ab} \equiv (A_a = +1, B_b = +1) \quad (2.0.1a)$$
$$(+1, -1)_{ab} \equiv (A_a = +1, B_b = -1) \quad (2.0.1b)$$
$$(-1, +1)_{ab} \equiv (A_a = -1, B_b = +1) \quad (2.0.1c)$$
$$(-1, -1)_{ab} \equiv (A_a = -1, B_b = -1) \quad (2.0.1d)$$

2.1 Boolean algebra for the case of one observable per site

2.1.1 Realism Hypothesis

In section 1.1 we talked about the Bohm-EPR experiment and the four simple events that it is possible to get in a single run of it when we only consider one measurement parameter per site (1.1.2). We have not said, however, at what point particles decide to adopt these values.

Let us assume that the values particles acquire are determined at the moment of particle's production and not at the measurement moment, in such a way that the information remained within them and would be there even if the measurement was never carried out. Under this assumption, it is possible to say that this information is a particles' inherent property, giving rise to the following hypothesis:

Hypothesis 3. ***Realism.*** *Particles emitted from the source carry within them the information to be measured. It is so, even if not measured.*

So, under this hypothesis, an *event* is the manifestation of a property that the pair of particles got at the time of production. In this way, A_a and B_b are the result of some inherent properties of the particles.

2.1.2 Important events

The objective of the Bohm-EPR experiment is to measure correlations, which implies measuring the properties of both particles simultaneously. Thereby, a simple, elementary or atomic event consists in the simultaneous measurement of A_a and B_b, which measured values are actually the properties gotten by the pair of particles at the time of production.

The four simple events for this experiment along with the value of the properties A_a and B_b measured on them are shown in Table 2.1.

Simple event	A_a	B_b
$G_1 = (+1, +1)_{ab}$	$+1$	$+1$
$G_2 = (+1, -1)_{ab}$	$+1$	-1
$G_3 = (-1, +1)_{ab}$	-1	$+1$
$G_4 = (-1, -1)_{ab}$	-1	-1

Table 2.1: Simple events and values of the properties A_a and B_b measured on them.

Each of these events corresponds to a function f assigned during production process, that goes from the pair of measurement options $s = (s_A, s_B)$ to the pair of possible results $v = (v_A, v_B)$ in such a way that:

$$f : s = (s_A, s_B) \longmapsto v = (v_A, v_B) \tag{2.1.1}$$

where $s_A = a$ and $s_B = b$ designate the measurement options at sites A and B, respectively, while that $v_A = \{+1, -1\}$ and $v_B = \{+1, -1\}$ are the possible values to be measured at sites A and B, respectively.

From these events it is possible to get the sample space for this experiment, denoted as Ω_{ab}:

$$\Omega_{ab} = \Big\{(+1,+1)_{ab},\ (+1,-1)_{ab},\ (-1,+1)_{ab},\ (-1,-1)_{ab}\Big\} \tag{2.1.2}$$

The boolean algebra of events (BAE) for this system is constructed from the four events of the sample space Ω_{ab}. For such a reason, we will denote this BAE as $\mathcal{B}(\Omega_{ab})$, which is an algebra of order 4, and it will have, therefore, $2^4 = 16$ events. These events are gotten by using the operation conjunction (AND, $\wedge$) and disjunction (OR, $\wedge$) over the elementary events of this algebra.

The properties of any BAE and the complete list of events of $\mathcal{B}(\Omega_{ab})$ are shown in the Appendix.

Like any BAE, the elementary events of $\mathcal{B}(\Omega_{ab})$ have the following properties:

$$
\begin{aligned}
&i) \qquad G_1 \vee G_2 \vee G_3 \vee G_4 = \mathbb{I}_{ab} && \text{(2.1.3a)}\\
&ii) \qquad G_i \wedge G_j = \begin{cases} G_i & \text{if } i = j, \text{ with } i,j = 1,2,3,4 \\ \varnothing_{ab} & \text{if } i \neq j, \text{ with } i,j = 1,2,3,4 \end{cases} && \text{(2.1.3b)}
\end{aligned}
$$

where $\mathbb{I}_{ab}$ and $\varnothing_{ab}$ are the ***certain*** and ***impossible events***, respectively, of $\mathcal{B}(\Omega_{ab})$.

Our interest is in measuring the properties particles carry with them. So, it is natural identifying events on which individual properties of *each particle* are measured. Such events receive the name of ***marginal events*** and, along with the individual property measured on them, which appears in parentheses, are presented below:

$$
\begin{aligned}
(+1, \mathbb{I}_b)_{ab} &\equiv (+1,+1)_{ab} \vee (+1,-1)_{ab} = G_1 \vee G_2 \qquad (A_a = +1) && \text{(2.1.4a)}\\
(-1, \mathbb{I}_b)_{ab} &\equiv (-1,+1)_{ab} \vee (-1,-1)_{ab} = G_3 \vee G_4 \qquad (A_a = -1) && \text{(2.1.4b)}\\
(\mathbb{I}_a, +1)_{ab} &\equiv (+1,+1)_{ab} \vee (-1,+1)_{ab} = G_1 \vee G_3 \qquad (B_b = +1) && \text{(2.1.4c)}\\
(\mathbb{I}_a, -1)_{ab} &\equiv (+1,-1)_{ab} \vee (-1,-1)_{ab} = G_2 \vee G_4 \qquad (B_b = -1) && \text{(2.1.4d)}
\end{aligned}
$$

However, what we are really interested in measuring is the correlation between particles. Therefore, it is more convenient to measure a property involving both particles.

So, from all the events of $\mathcal{B}(\Omega_{ab})$, only two of them are connected with the correlation between the pairs of particles since they tell us whether properties A_a and B_b have or not the same value. Such events (1.1.4) are the following:

$$
\begin{aligned}
\varepsilon_{ab} \equiv &(+1,+1)_{ab} \vee (-1,-1)_{ab} = G_1 \vee G_4 \qquad (A_a = +B_b) && \text{(2.1.5a)}\\
\delta_{ab} \equiv &(+1,-1)_{ab} \vee (-1,+1)_{ab} = G_2 \vee G_3 \qquad (A_a = -B_b) && \text{(2.1.5b)}
\end{aligned}
$$

Measuring correlations involves measuring properties from two particles. For this reason, it is more convenient to use the value of the product $A_a B_b$ as *property of the pair of particles* to measure during the experimental runs. So, it is possible to see that for the event ε_{ab} the product $A_a B_b$ takes the value $+1$ while for the event δ_{ab} it acquires the value -1.

Given that ε_{ab} and δ_{ab} are related with the value of the product $A_a B_b$, they will be named ***product events***.

This kind of event is what we are really interested in measuring during a experimental run because they help us to get the mean value of $A_a B_b$, which gives us, according to (1.1.6), the complete description of the correlations system.

An important property of the pair of product events $\{\varepsilon_{ab}, \delta_{ab}\}$ is that they are complementary events, i.e., $\delta'_{ab} = \varepsilon_{ab}$ and $\varepsilon'_{ab} = \delta_{ab}$. It can be easily verified by using (2.1.3) and (2.1.5).

In Table 2.2 we show the product events, their equivalence in atomic events and the value of the property $A_a B_b$ measured on them.

Product event	Equivalence in atomic events		$A_a B_b$
ε_{ab}	$(+1,+1)_{ab} \vee (-1,-1)_{ab}$	$G_1 \vee G_4$	$+1$
δ_{ab}	$(+1,-1)_{ab} \vee (-1,+1)_{ab}$	$G_2 \vee G_3$	-1

Table 2.2: Product events, their equivalence in atomic events and the value of the property $A_a B_b$ measured on them.

2.1.3 The probability algebra $< \mathcal{B}(\Omega_{ab}), P >$

The BAE $\mathcal{B}(\Omega_{ab})$, together with the Probability Function P, form a probability algebra, denoted as $< \mathcal{B}(\Omega_{ab}), P >$. A fuller explanation about probability algebras is given in Appendix C.

In such an algebra, a probability to happen is assigned to each of the events, specially to the atomic ones.

As we are interested in measuring correlations, we must find this occurrence probability for the product events ε_{ab} and δ_{ab}.

In this way:

$$\begin{aligned} i) \quad P\big(\varepsilon_{ab}\big) &= P\big((+1,+1)_{ab} \vee (-1,-1)_{ab}\big) \\ &= P\big((+1,+1)_{ab}\big) + P\big((-1,-1)_{ab}\big) \\ &= P\big(G_1\big) + P\big(G_4\big) \end{aligned} \tag{2.1.6}$$

and

$$\begin{aligned} ii) \quad P\big(\delta_{ab}\big) &= P\big((+1,-1)_{ab} \vee (-1,+1)_{ab}\big) \\ &= P\big((+1,-1)_{ab}\big) + P\big((-1,+1)_{ab}\big) \\ &= P\big(G_2\big) + P\big(G_3\big) \end{aligned} \tag{2.1.7}$$

where we have used in both cases (2.1.3b) and the properties of the Probability Function P.

Before finishing this section, it is important to say that thanks to the probability algebra $< \mathcal{B}(\Omega_{ab}), P >$ it is possible to find a realistic model for the ***experimental correlation*** M_{ab} (1.1.8) through the following definition:

Definition 3. *According to the probability algebra $< \mathcal{B}(\Omega_{ab}), P >$, the **Correlation Function** Q_{ab}, which predicts the value for the **experimental correlation** M_{ab}* (1.1.8)*, is given by:*

$$Q_{ab} = P\big(\varepsilon_{ab}\big) - P\big(\delta_{ab}\big) \tag{2.1.8}$$

Q_{ab} is, in this way, a prediction for the experimental value M_{ab} (1.1.8).

2.2 Boolean algebra for the case of two observables per site

2.2.1 Motivation

So far, we have just considered the experimental setup which parameters are a for site A and b for site B. However, we could equally change the parameters for such sites. In particular, we are interested in preparing an experimental setup such that the parameter for site A is still a, but now site B has switched its measurement parameter to c.

For this new experimental setup we have a new BAE, similar to $\mathcal{B}(\Omega_{ab})$, which sample space, denoted as Ω_{ac}, contains the following simple events:

$$\Omega_{ac} = \Big\{(+1,+1)_{ac},\ (+1,-1)_{ac},\ (-1,+1)_{ac},\ (-1,-1)_{ac}\Big\}\ . \tag{2.2.1}$$

We will refer to this new BAE as $\mathcal{B}(\Omega_{ac})$, which also forms a probability algebra $< \mathcal{B}(\Omega_{ac}), P >$. Both, $\mathcal{B}(\Omega_{ac})$ and $< \mathcal{B}(\Omega_{ac}), P >$, have the same properties that $\mathcal{B}(\Omega_{ab})$ and $< \mathcal{B}(\Omega_{ab}), P >$, respectively.

We now want to prepare a experimental setup with two configurations. In the first one, the parameter to carry out the measurements at site B is b, while in the second configuration is c. These configurations are, however, excluyent, i.e., we can perform measurements with b ***or*** c, but ***not with both in the same experimental run***. For such a reason, the measurement parameter to be used will be selected randomly in each experimental run.

In order to study such situation, we must construct a more general BAE apt to simultaneously describe both configurations, no matter if one of them is not measured. This new algebra must include any event from $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$, and even it should be reduced to these ones under the appropriate limits.

2.2.2 Construction of the algebra. Realism Hypothesis

As we have already said, we are interested in constructing an algebra apt to describe a system with two configurations, from which we can measure only in one of them, being impossible for us to know the information for the not-measured configuration.

Unlike the system having only one measurement parameter per site, where it was known *with certainty* what parameter would be used to measure in each site, in this new system it is impossible for the source that is emitting the particles to know what parameter will be selected at site B to perform the measurements given that the parameter to be used there will be chosen randomly. This leads us to extend the *Realism Hypothesis* stated above.

Hypothesis 4. ***Realism.*** *Particles emitted from the source must carry within the information to be measured for all the possible pairs of parameters to be selected at sites A and B. It is so, even if the information for one or more pairs of parameters is not measured.*

This hypothesis tells us that properties for every configuration are acquired during production process, and that these ones will be present within the particles even if not measured.

For such a reason, we look for an algebra which events are actually a list or set of properties, one for each system's configuration.

Given that both configurations are excluyent, an event simply is the manifestation of a property associated to the configuration where we perform the measurement.

Based on the above, we define the sample space of the algebra we are looking for in the following way:

Definition 4. ***Sample space*** Ω_{abc}. *Let* Ω_{ab} *and* Ω_{ac} *be the sample spaces for each configuration of the system separately. Then, the sample space for the system involving both configurations, denoted as* Ω_{abc}*, is defined as:*

$$\Omega_{abc} \equiv \Omega_{ab} \bigvee \Omega_{ac} = \Omega_{ac} \bigvee \Omega_{ab} = \Big\{ \{x_{ab},\, y_{ab}\} \Big| \, x_{ab} \in \Omega_{ab},\ y_{ac} \in \Omega_{ac} \Big\} \tag{2.2.2}$$

As $\{x_{ab},\, y_{ab}\}$ is only a grouping of properties, regardless the order of its elements, it turns out that $\{x_{ab},\, y_{ab}\} = \{y_{ab},\, x_{ab}\}$.

It is important to clarify that x_{ab} and y_{ac} are distinct properties because they refer to different configurations.

Once gotten the sample space Ω_{abc}, it is possible to construct the complete BAE for the 2-configuration system, which will be denoted as $\mathcal{B}(\Omega_{abc})$, by adding to Ω_{abc} the operations conjunction (AND, $\wedge$) and disjunction (OR, $\vee$). In addition to their usual properties (see Appendix A), these operations must comply with the following characteristics, which are derived directly from *Definition 4*:

1. **Distributivity of the conjunction:**

$$\Big\{x_{ab},\ y_{ac} \wedge z_{ac}\Big\} = \Big\{x_{ab},\ y_{ac}\Big\} \wedge \Big\{x_{ab},\ y_{ac}\Big\} \tag{2.2.3a}$$

$$\Big\{x_{ab} \wedge w_{ab},\ y_{ac}\Big\} = \Big\{x_{ab},\ y_{ac}\Big\} \wedge \Big\{w_{ab},\ y_{ac}\Big\} \tag{2.2.3b}$$

2. **Distributivity of the disjunction:**

$$\Big\{x_{ab},\ y_{ac} \vee z_{ac}\Big\} = \Big\{x_{ab},\ y_{ac}\Big\} \vee \Big\{x_{ab},\ z_{ac}\Big\} \tag{2.2.4a}$$

$$\Big\{x_{ab} \vee w_{ac},\ y_{ac}\Big\} = \Big\{x_{ab},\ y_{ac}\Big\} \vee \Big\{w_{ab},\ y_{ac}\Big\} \tag{2.2.4b}$$

where x_{ab} and $w_{ab} \in \Omega_{ab}$ while y_{ac} and $z_{ac} \in \Omega_{ac}$.

Given that Ω_{ab} and Ω_{ac} have each 4 simple events, Ω_{abc} will have 16 simple events. Consequently, $\mathcal{B}(\Omega_{abc})$ will have $2^{16} = 65,536$ events. Since it is impossible to analize such a number of events, we will only consider those ones useful to measure correlations. We dedicate subsections 2.2.5, 2.2.6 and 2.2.7 to study such events.

*(**Note:** As heretofore we have not assumed anything about locality yet, we have to take as feasible **all** those events. Later, restrictions imposed by Locality Hypothesis will make many of them to be the impossible event.)*

2.2.3 Atomic events

Let us now study the elementary events of $\mathcal{B}(\Omega_{abc})$. As we have already said before, this BAE has 16 elementary events, which are shown in Table 2.3.

Elementary events of $\mathcal{B}(\Omega_{abc})$	
$E_1 = \Big\{(+1,+1)_{ab},\ (+1,+1)_{ac}\Big\}$	$E_9 = \Big\{(-1,+1)_{ab},\ (+1,+1)_{ac}\Big\}$
$E_2 = \Big\{(+1,+1)_{ab},\ (+1,-1)_{ac}\Big\}$	$E_{10} = \Big\{(-1,+1)_{ab},\ (+1,-1)_{ac}\Big\}$
$E_3 = \Big\{(+1,+1)_{ab},\ (-1,+1)_{ac}\Big\}$	$E_{11} = \Big\{(-1,+1)_{ab},\ (-1,+1)_{ac}\Big\}$
$E_4 = \Big\{(+1,+1)_{ab},\ (-1,-1)_{ac}\Big\}$	$E_{12} = \Big\{(-1,+1)_{ab},\ (-1,-1)_{ac}\Big\}$
$E_5 = \Big\{(+1,-1)_{ab},\ (+1,+1)_{ac}\Big\}$	$E_{13} = \Big\{(-1,-1)_{ab},\ (+1,+1)_{ac}\Big\}$
$E_6 = \Big\{(+1,-1)_{ab},\ (+1,-1)_{ac}\Big\}$	$E_{14} = \Big\{(-1,-1)_{ab},\ (+1,-1)_{ac}\Big\}$
$E_7 = \Big\{(+1,-1)_{ab},\ (-1,+1)_{ac}\Big\}$	$E_{15} = \Big\{(-1,-1)_{ab},\ (-1,+1)_{ac}\Big\}$
$E_8 = \Big\{(+1,-1)_{ab},\ (-1,-1)_{ac}\Big\}$	$E_{16} = \Big\{(-1,-1)_{ab},\ (-1,-1)_{ac}\Big\}$

Table 2.3: The 16 elementary events of $\mathcal{B}(\Omega_{abc})$.

Like any BAE, atomic events of $\mathcal{B}(\Omega_{abc})$ have the following properties:

$$i) \qquad E_1 \vee \cdots \vee E_{16} = \mathbb{I}_{abc} \tag{2.2.5a}$$

$$ii) \qquad E_i \wedge E_j = \begin{cases} E_i & \text{if } i = j, \text{ with } i,j = 1,\ldots,16 \\ \varnothing_{abc} & \text{if } i \neq j, \text{ with } i,j = 1,\ldots,16 \end{cases} \tag{2.2.5b}$$

where $\mathbb{I}_{abc} \equiv \big(\mathbb{I}_{ab}, \mathbb{I}_{ac}\big)$ and $\varnothing_{abc} \equiv \big(\varnothing_{ab}, y_{ac}\big) \vee \big(x_{ab}, \varnothing_{ac}\big)$ are the ***certain*** and ***impossible events***, respectively, of $\mathcal{B}(\Omega_{abc})$.

These properties can be verified by using (2.2.3), (2.2.4) and the properties of $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$.

Using the notation introduced in Table 2.3, we can express Ω_{abc} as:

$$\Omega_{abc} = \big\{E_1, \cdots, E_{16}\big\} \tag{2.2.6}$$

On the other hand, *Realism Hypothesis* states that values to be measured at sites A and B for all possible pair of parameters are determined during production process. Under this hypothesis, each event from Table 2.3 corresponds to a function f determined during production process that goes from the set of measurement parameters $s = (s_A, s_B)$ to the set of measured values $v = (v_A, v_B)$, i.e.:

$$f : s = (s_A, s_B) \longmapsto v = (v_A(s), v_B(s)) \tag{2.2.7}$$

where $s_A = a$ and $s_B = \{b, c\}$ designate the measurement options at sites A and B, respectively, while that $v_A = \{+1, -1\}$ and $v_B = \{+1, -1\}$ are the measured values at sites A and B, respectively.

Since there are two pairs of parameters for s and four pairs of possible values for v, then there are $4^2 = 16$ distinct functions, which correspond to each of the 16 simple events of $\mathcal{B}(\Omega_{abc})$.

*(**Note:** Later, with the introduction of Locality Hypothesis, we will impose restrictions on f, in such a way that, under this hypothesis, the number of simple events decreases.)*

2.2.4 Sub-algebras $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$ in $\mathcal{B}(\Omega_{abc})$

In the previous subsection we constructed Ω_{abc} from Ω_{ab} and Ω_{ac}. Then, it should be possible to reduce $\mathcal{B}(\Omega_{abc})$ to $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$ in the appropriate limit.

We now show that it is possible to recover the simple events of $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$, and in consequence both algebras, from the elementary events of $\mathcal{B}(\Omega_{abc})$ (Table 2.3) and from equations (2.2.3) and (2.2.4).

The atomic events of $\mathcal{B}(\Omega_{ab})$ in $\mathcal{B}(\Omega_{abc})$

Let us begin finding the equivalence in $\mathcal{B}(\Omega_{abc})$ of the 4 simple events of $\mathcal{B}(\Omega_{ab})$.

$$\begin{aligned}
i)\;\; (+1,+1)_{ab} \equiv \Big\{(+1,+1)_{ab},\ \mathbb{I}_{ac}\Big\} &= \Big\{(+1,+1)_{ab},\ (+1,+1)_{ac} \vee (+1,-1)_{ac} \vee \\
&\qquad\qquad (-1,+1)_{ac} \vee (-1,-1)_{ac}\Big\} \\
&= \Big\{(+1,+1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,+1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,+1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,+1)_{ab},\ (-1,-1)_{ac}\Big\} \\
&= E_1 \vee E_2 \vee E_3 \vee E_4
\end{aligned} \tag{2.2.8}$$

where we have used the property (2.2.4a).

In a similar way:

$$
\begin{aligned}
ii)\ \ (+1,-1)_{ab} \equiv \Big\{(+1,-1)_{ab},\ \mathbb{I}_{ac}\Big\} &= \Big\{(+1,-1)_{ab},\ (+1,+1)_{ac} \vee (+1,-1)_{ac} \vee \\
&\qquad\qquad (-1,+1)_{ac} \vee (-1,-1)_{ac}\Big\} \\
&= \Big\{(+1,-1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,-1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,-1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,-1)_{ab},\ (-1,-1)_{ac}\Big\} \\
&= E_5 \vee E_6 \vee E_7 \vee E_8
\end{aligned}
\tag{2.2.9}
$$

$$
\begin{aligned}
iii)\ \ (-1,+1)_{ab} \equiv \Big\{(-1,+1)_{ab},\ \mathbb{I}_{ac}\Big\} &= \Big\{(-1,+1)_{ab},\ (+1,+1)_{ac} \vee (+1,-1)_{ac} \vee \\
&\qquad\qquad (-1,+1)_{ac} \vee (-1,-1)_{ac}\Big\} \\
&= \Big\{(-1,+1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,+1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,+1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,+1)_{ab},\ (-1,-1)_{ac}\Big\} \\
&= E_9 \vee E_{10} \vee E_{11} \vee E_{12}
\end{aligned}
\tag{2.2.10}
$$

$$
\begin{aligned}
iv)\ (-1,-1)_{ab} \equiv \Big\{(-1,-1)_{ab},\ \mathbb{I}_{ac}\Big\} &= \Big\{(-1,-1)_{ab},\ (+1,+1)_{ac} \vee (+1,-1)_{ac} \vee \\
&\qquad\qquad (-1,+1)_{ac} \vee (-1,-1)_{ac}\Big\} \\
&= \Big\{(-1,-1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,-1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,-1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,-1)_{ab},\ (-1,-1)_{ac}\Big\} \\
&= E_{13} \vee E_{14} \vee E_{15} \vee E_{16}
\end{aligned}
\tag{2.2.11}
$$

A summary of these equivalences is shown in Table 2.4.

The Atomic Events of $\mathcal{B}(\Omega_{ac})$ in $\mathcal{B}(\Omega_{abc})$

Let us now find the equivalence in $\mathcal{B}(\Omega_{abc})$ of the 4 Simple Events of $\mathcal{B}(\Omega_{ac})$.

Simple events of $\mathcal{B}(\Omega_{ab})$	Equivalence in $\mathcal{B}(\Omega_{abc})$	
$(+1,+1)_{ab}$	$\left\{(+1,+1)_{ab},\ \mathbb{I}_{ac}\right\}$	$E_1 \vee E_2 \vee E_3 \vee E_4$
$(+1,-1)_{ab}$	$\left\{(+1,-1)_{ab},\ \mathbb{I}_{ac}\right\}$	$E_5 \vee E_6 \vee E_7 \vee E_8$
$(-1,+1)_{ab}$	$\left\{(-1,+1)_{ab},\ \mathbb{I}_{ac}\right\}$	$E_9 \vee E_{10} \vee E_{11} \vee E_{12}$
$(-1,-1)_{ab}$	$\left\{(-1,-1)_{ab},\ \mathbb{I}_{ac}\right\}$	$E_{13} \vee E_{14} \vee E_{15} \vee E_{16}$

Table 2.4: Equivalence of the 4 atomic events of $\mathcal{B}(\Omega_{ab})$ in $\mathcal{B}(\Omega_{abc})$.

$$\begin{aligned}
i)\ \ (+1,+1)_{ac} \equiv \Big\{\mathbb{I}_{ab},\ (+1,+1)_{ac}\Big\} &= \Big\{(+1,+1)_{ab} \vee (+1,-1)_{ab} \vee \\
&\quad (-1,+1)_{ab} \vee (-1,-1)_{ab},\ (+1,+1)_{ac}\Big\} \\
&= \Big\{(+1,+1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,-1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,+1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,-1)_{ab},\ (+1,+1)_{ac}\Big\} \\
&= E_1 \vee E_5 \vee E_9 \vee E_{13}
\end{aligned} \tag{2.2.12}$$

where we have used the property (2.2.4b).

Analogously:

$$\begin{aligned}
ii)\ \ (+1,-1)_{ac} \equiv \Big\{\mathbb{I}_{ab},\ (+1,-1)_{ac}\Big\} &= \Big\{(+1,+1)_{ab} \vee (+1,-1)_{ab} \vee \\
&\quad (-1,+1)_{ab} \vee (-1,-1)_{ab},\ (+1,-1)_{ac}\Big\} \\
&= \Big\{(+1,+1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,-1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,+1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,-1)_{ab},\ (+1,-1)_{ac}\Big\} \\
&= E_2 \vee E_6 \vee E_{10} \vee E_{14}
\end{aligned} \tag{2.2.13}$$

$$
\begin{aligned}
iii)\ (-1,+1)_{ac} \equiv \Big\{\mathbb{I}_{ab},\ (-1,+1)_{ac}\Big\} &= \Big\{(+1,+1)_{ab} \vee (+1,-1)_{ab} \vee \\
&\qquad (-1,+1)_{ab} \vee (-1,-1)_{ab},\ (-1,+1)_{ac}\Big\} \\
&= \Big\{(+1,+1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,-1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,+1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,-1)_{ab},\ (-1,+1)_{ac}\Big\} \\
&= E_3 \vee E_7 \vee E_{11} \vee E_{15}
\end{aligned} \tag{2.2.14}
$$

$$
\begin{aligned}
iv)\ (-1,-1)_{ac} \equiv \Big\{\mathbb{I}_{ab},\ (-1,-1)_{ac}\Big\} &= \Big\{(+1,+1)_{ab} \vee (+1,-1)_{ab} \vee \\
&\qquad (-1,+1)_{ab} \vee (-1,-1)_{ab},\ (-1,-1)_{ac}\Big\} \\
&= \Big\{(+1,+1)_{ab},\ (-1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(+1,-1)_{ab},\ (-1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,+1)_{ab},\ (-1,-1)_{ac}\Big\} \vee \\
&\quad \Big\{(-1,-1)_{ab},\ (-1,-1)_{ac}\Big\} \\
&= E_4 \vee E_8 \vee E_{12} \vee E_{16}
\end{aligned} \tag{2.2.15}
$$

These equivalences are summarized in Table 2.5.

Simple events of $\mathcal{B}(\Omega_{ac})$	**Equivalence in $\mathcal{B}(\Omega_{abc})$**	
$(+1,+1)_{ac}$	$\Big\{\mathbb{I}_{ab},\ (+1,+1)_{ac}\Big\}$	$E_1 \vee E_5 \vee E_9 \vee E_{13}$
$(+1,-1)_{ac}$	$\Big\{\mathbb{I}_{ab},\ (+1,-1)_{ac}\Big\}$	$E_2 \vee E_6 \vee E_{10} \vee E_{14}$
$(-1,+1)_{ac}$	$\Big\{\mathbb{I}_{ab},\ (-1,+1)_{ac}\Big\}$	$E_3 \vee E_7 \vee E_{11} \vee E_{15}$
$(-1,-1)_{ac}$	$\Big\{\mathbb{I}_{ab},\ (-1,-1)_{ac}\Big\}$	$E_4 \vee E_8 \vee E_{12} \vee E_{16}$

Table 2.5: Equivalence of the 4 simple events of $\mathcal{B}(\Omega_{ac})$ in $\mathcal{B}(\Omega_{abc})$.

Once known the equivalence of the atomic events of $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$ in $\mathcal{B}(\Omega_{abc})$, it is possible to find the equivalence in $\mathcal{B}(\Omega_{abc})$ of any event from $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$.

2.2.5 Correlation events

In subsection 2.1.2 we saw that it is more useful to measure the product A_aB_b that the individual properties A_a and B_b when we look for measuring correlations between pairs of particles because A_aB_b involves properties from both of them. We also said that in order to completely characterize the correlations system, it was enough to determine the value of M_{ab}, i.e., the mean value of A_aB_b (1.1.6).

We now have a system with two excluyent configurations to perform the measurements. In the configuration with measurement parameter b for site B, the product to determine is still A_aB_b; however, for the other configuration, whose measurement parameter for site B is c, the product to be determined is A_aB_c. For such a reason, we now need a greater amount of information to completely characterize the correlations system.

In particular, along with the mean value of the two products mentioned above, we need to determine the mean value of the product B_bB_c, which involves information related with the properties of the particle located at site B for both configurations.

However, there is a little problem: it is not possible to measure the three products in a same experimental run given that the two configurations are excluyent.

Nevertheless, according to ***Realism Hypothesis*** *(Hypothesis 4) "particles emitted from the source carry within them the information to be measured for all possible pairs of parameters to be selected at sites A and B; it is so, even if the information for one or more pairs of parameters is not measured"*. Therefore, we can assume that the information for A_aB_b, A_aB_c and B_bB_c, which is related to some inherent properties of the particles, is already contained within them, even if we are not able to measure for all the pairs of parameters in a same experimental run.

We must now identify the events of $\mathcal{B}(\Omega_{abc})$ producing those three products.

To obtain the value of A_aB_b, A_aB_c and B_bB_c from each elementary event of Table 2.3, we must multiply:

1. The two values of the first parentheses, which refers to the configuration with parameters a and b, to get A_aB_b.

2. The two values of the second parentheses, which refers to the configuration with parameters a and c, to get A_aB_c.

3. The second value of both parentheses, which involves both configurations, to get B_bB_c.

Once knowing the value of A_aB_b, A_aB_c and B_bB_c for each atomic event, it is convenient to define a new kind of event from the disjunction of the elementary events with the same value for each of these products. These new events will be named ***correlation events*** because they provide to each single run of a complete description of the correlation between the pair of particles, i.e., they contain the value for the products A_aB_b, A_aB_c and B_bB_c for each single run.

Given that these products can each take two values ($+1$ and -1), it is possible to form 8 events of this kind. These events are shown in the Table 2.6. There it is possible to see the value of each product for the 8 correlation events.

Correlation events comply with the following properties:

Correlation events	**Equivalence in atomic events**	A_aB_b	A_aB_c	B_bB_c
C_1	$E_1 \vee E_{16}$	$+1$	$+1$	$+1$
C_2	$E_4 \vee E_{13}$	$+1$	$+1$	-1
C_3	$E_3 \vee E_{14}$	$+1$	-1	$+1$
C_4	$E_2 \vee E_{15}$	$+1$	-1	-1
C_5	$E_8 \vee E_9$	-1	$+1$	$+1$
C_6	$E_5 \vee E_{12}$	-1	$+1$	-1
C_7	$E_6 \vee E_{11}$	-1	-1	$+1$
C_8	$E_7 \vee E_{10}$	-1	-1	-1

Table 2.6: Definition of the correlation events, their equivalence in elementary events of $\mathcal{B}(\Omega_{abc})$ and their value for the products A_aB_b, A_aB_c and B_bB_c.

$$i) \qquad C_1 \vee \cdots \vee C_8 = \mathbb{I}_{abc} \tag{2.2.16a}$$

$$ii) \qquad C_i \wedge C_j = \begin{cases} C_i & \text{if } i = j, \text{ with } i,j = 1,\ldots,8 \\ \varnothing_{abc} & \text{if } i \neq j, \text{ with } i,j = 1,\ldots,8 \end{cases} \tag{2.2.16b}$$

These properties can be verified by using (2.2.5) and the values from the Table 2.6.

Since correlation events contain the information for each product and they have the properties mentioned above (2.2.16), they are a useful set of events to describe the correlations between the pairs of particles. For such a reason, from now on we will express everything in terms of these events.

It is important to remember that we have not said anything about locality yet. Later on we will see that with the introduction of this hypothesis, several correlation events will be the impossible event.

2.2.6 Product events and the Correlation Function

In the previous subsection we said that in order to completely characterize the correlations system between the pairs of particles we needed to determine the mean value of A_aB_b, A_aB_c and B_bB_c. For this reason, we defined the correlation events, which contain the values for each of these products to be measured in a *single run*.

However, given that the two configurations of the experimental setup are excluyent, it is impossible for us to measure in a *same* run the value of the three products.

Therefore, we need to find a new kind of event that *only* contains the information that we are able to *completely* measure in a single experimental run, i.e., we must look for events containing *only* the information for *one* of those products. To do this, we will follow a similar way to that from section 2.1.2.

In that section we looked for determining the mean value of A_aB_b, so we found convenient to define the events ε_{ab} and δ_{ab} based on the value that this product took in each measurement: the event ε_{ab} considered the cases where A_a and B_b got the same value (i.e.,

$A_aB_b = +1$), while δ_{ab} took into account the cases where A_a and B_b acquired a different value (i.e., $A_aB_b = -1$).

We named ***product events*** to ε_{ab} and δ_{ab} because they were defined based on the value that the product A_aB_b get when measured. We then said that this kind of event was what we must measure in the lab.

Finally, we said that the Correlation Function Q_{ab}, which is the classical prediction for the mean value of A_aB_b, depended only on the events ε_{ab} and δ_{ab} (2.1.8).

Something similar happens with the configuration with measurement parameter c: the product A_aB_c has as Correlation Function to Q_{ac}, which depends, in turn, on the product events ε_{ac} ($A_aB_c = +1$) and δ_{ac} ($A_aB_c = -1$).

Then, we see that product events contain the information for only *one* product, regardless the information for the other products.

Let us now come back to the 2-configuration system where we were working on. Such system considers the two configurations described above.

Product events for $\mathcal{B}(\Omega_{abc})$ must be similar to those developed for $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$, thus the only thing we need to do is to express ε_{ab}, δ_{ab}, ε_{ac} and δ_{ac} in terms of common algebra $\mathcal{B}(\Omega_{abc})$, which contains the information for both configurations. Therefore, from Tables 2.4, 2.5, 2.6, we have that:

$$\begin{aligned} i) \quad \varepsilon_{ab} \equiv \Big\{\varepsilon_{ab},\ \mathbb{I}_{ac}\Big\} &= \Big\{(+1,+1)_{ab} \vee (-1,-1)_{ab},\ \mathbb{I}_{ac}\Big\} \\ &= \Big\{(+1,+1)_{ab},\ \mathbb{I}_{ac}\Big\} \vee \Big\{(-1,-1)_{ab},\ \mathbb{I}_{ac}\Big\} \\ &= \Big\{E_1 \vee E_2 \vee E_3 \vee E_4\Big\} \vee \Big\{E_{13} \vee E_{14} \vee E_{15} \vee E_{16}\Big\} \\ &= E_1 \vee E_2 \vee E_3 \vee E_4 \vee E_{13} \vee E_{14} \vee E_{15} \vee E_{16} \\ &= C_1 \vee C_2 \vee C_3 \vee C_4 \end{aligned} \tag{2.2.17}$$

$$\begin{aligned} ii) \quad \delta_{ab} \equiv \Big\{\delta_{ab},\ \mathbb{I}_{ac}\Big\} &= \Big\{(+1,-1)_{ab} \vee (-1,+1)_{ab},\ \mathbb{I}_{ac}\Big\} \\ &= \Big\{(+1,-1)_{ab},\ \mathbb{I}_{ac}\Big\} \vee \Big\{(-1,+1)_{ab},\ \mathbb{I}_{ac}\Big\} \\ &= \Big\{E_5 \vee E_6 \vee E_7 \vee E_8\Big\} \vee \Big\{E_9 \vee E_{10} \vee E_{11} \vee E_{12}\Big\} \\ &= E_5 \vee E_6 \vee E_7 \vee E_8 \vee E_9 \vee E_{10} \vee E_{11} \vee E_{12} \\ &= C_5 \vee C_6 \vee C_7 \vee C_8 \end{aligned} \tag{2.2.18}$$

$$\begin{aligned} iii) \quad \varepsilon_{ac} \equiv \Big\{\mathbb{I}_{ab},\ \varepsilon_{ac}\Big\} &= \Big\{\mathbb{I}_{ab}, (+1,+1)_{ac} \vee (-1,-1)_{ac}\Big\} \\ &= \Big\{\mathbb{I}_{ab},\ (+1,+1)_{ac}\Big\} \vee \Big\{\mathbb{I}_{ab},\ (-1,-1)_{ac}\Big\} \\ &= \Big\{E_1 \vee E_5 \vee E_9 \vee E_{13}\Big\} \vee \Big\{E_4 \vee E_8 \vee E_{12} \vee E_{16}\Big\} \\ &= E_1 \vee E_4 \vee E_5 \vee E_8 \vee E_9 \vee E_{12} \vee E_{13} \vee E_{16} \\ &= C_1 \vee C_2 \vee C_5 \vee C_6 \end{aligned} \tag{2.2.19}$$

$$
\begin{aligned}
iv) \quad \delta_{ac} \equiv \Big\{\mathbb{I}_{ab},\ \delta_{ac}\Big\} &= \Big\{\mathbb{I}_{ab}, (+1,-1)_{ac} \vee (-1,+1)_{ac}\Big\} \\
&= \Big\{\mathbb{I}_{ab},\ (+1,-1)_{ac}\Big\} \vee \Big\{\mathbb{I}_{ab},\ (-1,+1)_{ac}\Big\} \\
&= \Big\{E_2 \vee E_6 \vee E_{10} \vee E_{14}\Big\} \vee \Big\{E_3 \vee E_7 \vee E_{11} \vee E_{15}\Big\} \qquad (2.2.20) \\
&= E_2 \vee E_3 \vee E_6 \vee E_7 \vee E_{10} \vee E_{11} \vee E_{14} \vee E_{15} \\
&= C_3 \vee C_4 \vee C_7 \vee C_8
\end{aligned}
$$

where the terms to the left of the equivalence sign are events of $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$, while the terms to the right are events of $\mathcal{B}(\Omega_{abc})$.

To make it simple, it is convenient to use a new nomenclature for the product events described above:

$$U_1 \equiv \Big\{\varepsilon_{ab},\ \mathbb{I}_{ac}\Big\} \qquad (2.2.21a)$$

$$U_2 \equiv \Big\{\delta_{ab},\ \mathbb{I}_{ac}\Big\} \qquad (2.2.21b)$$

and

$$V_1 \equiv \Big\{\mathbb{I}_{ab},\ \varepsilon_{ac}\Big\} \qquad (2.2.22a)$$

$$V_2 \equiv \Big\{\mathbb{I}_{ab},\ \delta_{ac}\Big\} \qquad (2.2.22b)$$

where $\{U_1, U_2\}$ and $\{V_1, V_2\} \subset \mathcal{B}(\Omega_{abc})$.

In the same way as we did in subsection 2.1.2, it is convenient to group by pairs the product events $\{U_1, U_2\}$ and $\{V_1, V_2\}$ because they are complementary events, i.e., $U_2 = U_1'$, $U_1 = U_2'$, $V_2 = V_1'$ and $V_1 = V_2'$. It can be easily verified by using (2.2.17) - (2.2.20) and (2.2.16). We will see the usefulness of this property later in Chapter 3.

The importance of the product events lies on that they are what we are interested in measuring in the lab.

Let us now define the probability algebra $< \mathcal{B}(\Omega_{abc}),\ P >$ in a similar way as we did in subsection 2.1.3. Such an algebra assigns a probability to happen to each event of $\mathcal{B}(\Omega_{abc})$.

We are particularly interested in assigning such probability to the product events because the Correlation Functions Q_{ab} and Q_{ac} only depend on them. In this way, we can now express (2.1.8), which is defined in $< \mathcal{B}(\Omega_{ab}),\ P >$, in terms of $< \mathcal{B}(\Omega_{abc}),\ P >$.

Then:

$$Q_{ab} = P(\varepsilon_{ab}) - P(\delta_{ab}) \equiv P(U_1) - P(U_2)\ , \qquad (2.2.23)$$

where ε_{ab} and $\delta_{ab} \in \mathcal{B}(\Omega_{ab})$, while U_1 and $U_2 \in \mathcal{B}(\Omega_{abc})$.

In a similar way:

$$Q_{ac} = P(\varepsilon_{ac}) - P(\delta_{ac}) \equiv P(V_1) - P(V_2) \ , \tag{2.2.24}$$

where ε_{ac} and $\delta_{ac} \in \mathcal{B}(\Omega_{ac})$, while V_1 and $V_2 \in \mathcal{B}(\Omega_{abc})$.

At this point, it is possible to think that, as well as $A_a B_b$ and $A_a B_c$ have a Correlation Function and they are associated to specific pairs of product events, $B_b B_c$ must be too. However, there is a little difference.

While $A_a B_b$ and $A_a B_c$ involve information about only one configuration and they can be measured in a same run, $B_b B_c$ involves information about the two configurations and, as a consequence, it cannot be measured in a same run.

For this reason, it is precise to define a new pair of product events and a new Correlation Function for $B_b B_c$.

Definition 5. ***Product events*** ξ_{bc} ***and*** η_{bc}.

1. *The* ***product event*** ξ_{bc} *is the compound event formed by the disjunction of all the simple events of* Ω_{abc} *such that* $B_b = B_c$*, i.e.:*

$$\begin{aligned}
\xi_{bc} = &\Big\{(+1,+1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \Big\{(+1,+1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\Big\{(+1,-1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \Big\{(+1,-1)_{ab},\ (-1,-1)_{ac}\Big\} \vee \\
&\Big\{(-1,+1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \Big\{(-1,+1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\Big\{(-1,-1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \Big\{(-1,-1)_{ab},\ (-1,-1)_{ac}\Big\} \\
= &\ E_1 \vee E_3 \vee E_6 \vee E_8 \vee E_9 \vee E_{11} \vee E_{14} \vee E_{16} \\
= &\ C_1 \vee C_3 \vee C_5 \vee C_7
\end{aligned} \tag{2.2.25}$$

2. *The* ***product event*** η_{bc} *is the compound event formed by the disjunction of all the simple events of* Ω_{abc} *such that* $B_b = -B_c$*, i.e.:*

$$\begin{aligned}
\eta_{bc} = &\Big\{(+1,+1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \Big\{(+1,+1)_{ab},\ (-1,-1)_{ac}\Big\} \vee \\
&\Big\{(+1,-1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \Big\{(+1,-1)_{ab},\ (-1,+1)_{ac}\Big\} \vee \\
&\Big\{(-1,+1)_{ab},\ (+1,-1)_{ac}\Big\} \vee \Big\{(-1,+1)_{ab},\ (-1,-1)_{ac}\Big\} \vee \\
&\Big\{(-1,-1)_{ab},\ (+1,+1)_{ac}\Big\} \vee \Big\{(-1,-1)_{ab},\ (-1,+1)_{ac}\Big\} \\
= &\ E_2 \vee E_4 \vee E_5 \vee E_7 \vee E_{10} \vee E_{12} \vee E_{13} \vee E_{15} \\
= &\ C_2 \vee C_4 \vee C_6 \vee C_8
\end{aligned} \tag{2.2.26}$$

Let us now state a useful lemma about the symmetry of these events in relation to the interchange of the parameters b and c.

Lemma 2. *The events ξ_{bc} and η_{bc} are symmetric in relation to the interchange of subscripts, i.e.:*

$$\xi_{bc} = \xi_{cb} \tag{2.2.27a}$$
$$\eta_{bc} = \eta_{cb} \tag{2.2.27b}$$

Proof. It is a direct consequence from *Definitions 4 and 5.* □

Once defined ξ_{bc} and η_{bc}, it is possible to obtain the Correlation Function for the product $B_b B_c$. This new function must be similar to the Correlation Function Q because they are classical predictions for the mean value of the product of two properties.

Definition 6. *The **Correlation Function** D is defined as the classical prediction for the mean value of the product $B_b B_c$. This function only depends on the events ξ_{bc} and η_{bc}, and is given by:*

$$D_{bc} \equiv P\big(\xi_{bc}\big) - P\big(\eta_{bc}\big) \tag{2.2.28}$$

Given that ξ_{bc} and η_{bc} are symmetric in relation to the interchange of parameters b and c, the Correlation Function D is too. It is shown in the following corollary:

Corollary 1. *The Correlation Function D is symmetric in relation to the interchange of subscripts, i.e.:*

$$D_{bc} = D_{cb} \tag{2.2.29}$$

Proof.

$$\begin{aligned} D_{bc} &= P(\xi_{bc}) - P(\eta_{bc}) \\ &= P(\xi_{cb}) - P(\eta_{cb}) \\ &= D_{cb}\ , \end{aligned} \tag{2.2.30}$$

where we have used *Lemma 2* to get the second line. □

As $\{U_1, U_2\}$ and $\{V_1, V_2\}$, it is also convenient to group in a pair the events $\{\xi_{bc}, \eta_{bc}\}$ because they are complementary events too, i.e., $\eta_{bc} = \xi'_{bc}$ and $\xi_{bc} = \eta'_{bc}$. It can be easily verified by using (2.2.25), (2.2.26) and (2.2.16).

The fact that the pairs $\{U_1, U_2\}$, $\{V_1, V_2\}$ and $\{\xi_{bc}, \eta_{bc}\}$ are complementary events is a handy property that will be used later in Chapter 3 to obtain the Bell-CHSH inequality in a merely algebraic way.

The product events, their equivalence in $\mathcal{B}(\Omega_{abc})$ and the products they are related to are shown in Table 2.7.

Events in $\mathcal{B}(\Omega_{ab})$ and $\mathcal{B}(\Omega_{ac})$	Equivalence in $\mathcal{B}(\Omega_{abc})$			Product measured and its value
ε_{ab}	$\left\{\varepsilon_{ab},\ \mathbb{I}_{ac}\right\}$	U_1	$C_1 \vee C_2 \vee C_3 \vee C_4$	$A_a B_b = +1$
δ_{ab}	$\left\{\delta_{ab},\ \mathbb{I}_{ac}\right\}$	U_2	$C_5 \vee C_6 \vee C_7 \vee C_8$	$A_a B_b = -1$
ε_{ac}	$\left\{\mathbb{I}_{ab},\ \varepsilon_{ac}\right\}$	V_1	$C_1 \vee C_2 \vee C_5 \vee C_6$	$A_a B_c = +1$
δ_{ac}	$\left\{\mathbb{I}_{ab},\ \delta_{ac}\right\}$	V_2	$C_3 \vee C_4 \vee C_7 \vee C_8$	$A_a B_c = -1$
———	ξ_{bc}		$C_1 \vee C_3 \vee C_5 \vee C_7$	$B_b B_c = +1$
———	η_{bc}		$C_2 \vee C_4 \vee C_6 \vee C_8$	$B_b B_c = -1$

Table 2.7: The product events, their equivalence in $\mathcal{B}(\Omega_{abc})$ and the products they are related to.

2.2.7 Events defined from two conditions

In previous subsections we have already studied events defined on 3 conditions (correlation events) and over 1 condition (product events). The first ones provide a complete description of the correlations system for each *single* run, while the second ones only contain the information we are able to measure in a *single* run.

Let us now study events defined on 2 products or conditions. In order to obtain them, we will use the conjunction of product events.

1. **Events obtained from pairs** $\{U_1, U_2\}$ **and** $\{V_1, V_2\}$**:** (conditions on $A_a B_b$ and $A_a B_c$)

$$\begin{aligned} i) \quad U_1 \wedge V_1 &= \Big(C_1 \vee C_2 \vee C_3 \vee C_4\Big) \wedge \Big(C_1 \vee C_2 \vee C_5 \vee C_6\Big) \\ &= C_1 \vee C_2 \end{aligned} \tag{2.2.31}$$

$$\begin{aligned} ii) \quad U_1 \wedge V_2 &= \Big(C_1 \vee C_2 \vee C_3 \vee C_4\Big) \wedge \Big(C_3 \vee C_4 \vee C_7 \vee C_8\Big) \\ &= C_3 \vee C_4 \end{aligned} \tag{2.2.32}$$

$$\begin{aligned} iii) \quad U_2 \wedge V_1 &= \Big(C_5 \vee C_6 \vee C_7 \vee C_8\Big) \wedge \Big(C_1 \vee C_2 \vee C_5 \vee C_6\Big) \\ &= C_5 \vee C_6 \end{aligned} \tag{2.2.33}$$

$$\begin{aligned} iv) \quad u_2 \wedge v_2 &= \Big(C_5 \vee C_6 \vee C_7 \vee C_8\Big) \wedge \Big(C_3 \vee C_4 \vee C_7 \vee C_8\Big) \\ &= C_7 \vee C_8 \end{aligned} \tag{2.2.34}$$

2. **Events obtained from pairs** $\{U_1, U_2\}$ **and** $\{\xi_{bc}, \eta_{bc}\}$**:** (conditions on $A_a B_b$ and $B_b B_c$)

$$\begin{aligned} i) \quad U_1 \wedge \xi_{bc} &= \Big(C_1 \vee C_2 \vee C_3 \vee C_4\Big) \wedge \Big(C_1 \vee C_3 \vee C_5 \vee C_7\Big) \\ &= C_1 \vee C_3 \end{aligned} \tag{2.2.35}$$

$$ii)\quad U_1 \wedge \eta_{bc} = \Big(C_1 \vee C_2 \vee C_3 \vee C_4\Big) \wedge \Big(C_2 \vee C_4 \vee C_6 \vee C_8\Big) \\ = C_2 \vee C_4 \tag{2.2.36}$$

$$iii)\quad U_2 \wedge \xi_{bc} = \Big(C_5 \vee C_6 \vee C_7 \vee C_8\Big) \wedge \Big(C_1 \vee C_3 \vee C_5 \vee C_7\Big) \\ = C_5 \vee C_7 \tag{2.2.37}$$

$$iv)\quad U_2 \wedge \eta_{bc} = \Big(C_5 \vee C_6 \vee C_7 \vee C_8\Big) \wedge \Big(C_2 \vee C_4 \vee C_6 \vee C_8\Big) \\ = C_6 \vee C_8 \tag{2.2.38}$$

3. **Events obtained from pairs $\{V_1, V_2\}$ and $\{\xi_{bc}, \eta_{bc}\}$:** (conditions on $A_a B_c$ and $B_b B_c$)

$$i)\quad V_1 \wedge \xi_{bc} = \Big(C_1 \vee C_2 \vee C_5 \vee C_6\Big) \wedge \Big(C_1 \vee C_3 \vee C_5 \vee C_7\Big) \\ = C_1 \vee C_5 \tag{2.2.39}$$

$$ii)\quad V_1 \wedge \eta_{bc} = \Big(C_1 \vee C_2 \vee C_5 \vee C_6\Big) \wedge \Big(C_2 \vee C_4 \vee C_6 \vee C_8\Big) \\ = C_2 \vee C_6 \tag{2.2.40}$$

$$iii)\quad V_2 \wedge \xi_{bc} = \Big(C_3 \vee C_4 \vee C_7 \vee C_8\Big) \wedge \Big(C_1 \vee C_3 \vee C_5 \vee C_7\Big) \\ = C_3 \vee C_7 \tag{2.2.41}$$

$$iv)\quad V_2 \wedge \eta_{bc} = \Big(C_3 \vee C_4 \vee C_7 \vee C_8\Big) \wedge \Big(C_2 \vee C_4 \vee C_6 \vee C_8\Big) \\ = C_4 \vee C_8 \tag{2.2.42}$$

These events are summarized in Table 2.8. Their importance will be seen later on with the introduction of *Locality Hypothesis*.

2.2.8 Locality Hypothesis

As we have already said, *Realism Hypothesis* proposes that particles emitted from the source carry within the information to be measured for all possible pairs of parameters to be selected at sites A and B. This hypothesis, however, does not place any kind of restriction on the information carried within the particles beyond the fact that this one is produced by pairs, one for each system's configuration.

$\wedge$	U_1	U_2	V_1	V_2	ξ_{bc}	η_{bc}
U_1	U_1	$\varnothing_{abc}$	$C_1 \vee C_2$	$C_3 \vee C_4$	$C_1 \vee C_3$	$C_2 \vee C_4$
U_2	$\varnothing_{abc}$	U_2	$C_5 \vee C_6$	$C_7 \vee C_8$	$C_5 \vee C_7$	$C_6 \vee C_8$
V_1	$C_1 \vee C_2$	$C_5 \vee C_6$	V_1	$\varnothing_{abc}$	$C_1 \vee C_5$	$C_2 \vee C_6$
V_2	$C_3 \vee C_4$	$C_7 \vee C_8$	$\varnothing_{abc}$	V_2	$C_3 \vee C_7$	$C_4 \vee C_8$
ξ_{bc}	$C_1 \vee C_3$	$C_5 \vee C_7$	$C_1 \vee C_5$	$C_3 \vee C_7$	ξ_{bc}	$\varnothing_{abc}$
η_{bc}	$C_2 \vee C_4$	$C_6 \vee C_8$	$C_2 \vee C_6$	$C_4 \vee C_8$	$\varnothing_{abc}$	η_{bc}

Table 2.8: Events defined on two conditions. Given that each product event is related to the value of one product, then the conjunction of two of them contains information on the value of two products.

The absence of an additional restriction allows, for example, cases where A_a takes a different value depending on the measurement option chosen at site B, which implies instant and long-distance communication, specially if sites A and B are located far away from each other.

Given that we want to reject this kind of communication from the problem's description, we must add any restriction on information so that cases like the stated above are prohibited. Thus, we propose the following hypothesis:

Hypothesis 5. ***Locality.*** *Sites A and B are arbitrarily far away from each other and from the source that is emitting the particles, and any kind of instant and long-distance communication between them has been rejected. Then, measurements made at site A have* ***no effect*** *on measurements carried out at site B, and vice-versa. In particular, switching the measurement parameter at site B does not affect the result at site A.*

The hypothesis above implies that, although the information was produced by pairs during production process, each particle carries within *only* those properties assigned to it, regardless of the properties laid on the other particle. Then, it is possible to "factorize" a *unique* value for A_a, B_b and B_c from the information contained within the simple events.

Therefore, under locality there are independent functions v_A and v_B such that $v_A : a \longmapsto \{+1, -1\}$ at site A and $v_B : \{b, c\} \longmapsto \{+1, -1\}$ at site B, which enables that function f (2.2.7) can be *factorized* in the following way:

$$f : s = (s_A, s_B) \longmapsto v = (v_A(s_A), v_B(s_B)) \tag{2.2.43}$$

Then, function v_A corresponds to the property A_a, while function v_B corresponds to the properties measured at site B.

Each function f corresponds to one atomic event allowed by *Locality Hypothesis*. Then, since there are 2 v_A and 4 v_B different functions, the total number of simple events under locality is $2 \times 4 = 8$. These events along with their *factorized* values for A_a, B_b and B_c are shown in Table 2.9.

In this way, factorization is the condition imposed by locality on the algebra of events.

On the other hand, simple events with no factorizable function f as given in (2.2.43) are prohibited; in other words, those events for which it is not possible to factorize a *unique* value for A_a, B_b and B_c from the information contained within them are the impossible event $\varnothing_{abc}$. It can be expressed in the following way:

Atomic event		A_a	B_b	B_c
E_1	$\Big\{(+1,+1)_{ab},\ (+1,+1)_{ac}\Big\}$	$+1$	$+1$	$+1$
E_2	$\Big\{(+1,+1)_{ab},\ (+1,-1)_{ac}\Big\}$	$+1$	$+1$	-1
E_5	$\Big\{(+1,-1)_{ab},\ (+1,+1)_{ac}\Big\}$	$+1$	-1	$+1$
E_6	$\Big\{(+1,-1)_{ab},\ (+1,-1)_{ac}\Big\}$	$+1$	-1	-1
E_{11}	$\Big\{(-1,+1)_{ab},\ (-1,+1)_{ac}\Big\}$	-1	$+1$	$+1$
E_{12}	$\Big\{(-1,+1)_{ab},\ (-1,-1)_{ac}\Big\}$	-1	$+1$	-1
E_{15}	$\Big\{(-1,-1)_{ab},\ (-1,+1)_{ac}\Big\}$	-1	-1	$+1$
E_{16}	$\Big\{(-1,-1)_{ab},\ (-1,-1)_{ac}\Big\}$	-1	-1	-1

Table 2.9: Atomic events allowed by locality along with their factorized values for A_a, B_b and B_c.

$$\Big\{(\pm 1, x)_{ab},\ (\mp 1, y)_{ac}\Big\} = \varnothing_{abc}\ ,\ \text{where } x, y = \{+1, -1\}\ . \tag{2.2.44}$$

So, under locality, atomic events E_3, E_4, E_7, E_8, E_9, E_{10}, E_{13} and E_{14} are the impossible event $\varnothing_{abc}$.

Locality Hypothesis drastically reduces the number of events allowed by the single *Realism Hypothesis* since the number of atomic events is reduced from 16 to 8, going from an algebra of $2^{16} = 65,536$ events to one of only $2^8 = 256$ events.

Moreover, with the introduction of *Locality Hypothesis* and the condition of factorization, we are also imposing a restriction on the values that A_aB_b, A_aB_c and B_bB_c can take. Thus, we obtain the following *Locality Criterion for products*:

$$A_aB_b \times A_aB_c = B_bB_c \tag{2.2.45}$$

This is also a *Locality Criterion for correlation events* since these ones are based on the value each of the three products takes for a single run.

Correlation events along with their value for each product are shown in Table 2.10. If the event complies with the property of factorization imposed by the *Locality Hypothesis* (2.2.43) or (2.2.45), such an event is a local event (L); otherwise, the event is non-local (NL), being, therefore, the impossible event $\varnothing_{abc}$.

From this table it is possible to see that the non-local character of correlation events C_2, C_3, C_5 and C_8 can be obtained by using either (2.2.44) or (2.2.45).

As under locality many atomic and correlation events are the impossible event $\varnothing_{abc}$, product events are simplified. It is shown in Table 2.11.

Let us now discuss the importance of *Locality Hypothesis*.

In Table 2.8 events defined on 2 conditions were shown. With the simplifications introduced by locality, events from that table get smaller. This is shown in Table 2.12.

Correlation events	**Equivalence in atomic events**	A_aB_b	A_aB_c	B_bB_c	**Locality**
C_1	$E_1 \vee E_{16}$	$+1$	$+1$	$+1$	L
C_2	$E_4 \vee E_{13}$	$+1$	$+1$	-1	NL
C_3	$E_3 \vee E_{14}$	$+1$	-1	$+1$	NL
C_4	$E_2 \vee E_{15}$	$+1$	-1	-1	L
C_5	$E_8 \vee E_9$	-1	$+1$	$+1$	NL
C_6	$E_5 \vee E_{12}$	-1	$+1$	-1	L
C_7	$E_6 \vee E_{11}$	-1	-1	$+1$	L
C_8	$E_7 \vee E_{10}$	-1	-1	-1	NL

Table 2.10: Local (L) and non-local (NL) correlation events.

Product event	**Equivalence in atomic events**	**Equivalence in correlation events**
U_1	$E_1 \vee E_2 \vee E_{15} \vee E_{16}$	$C_1 \vee C_4$
U_2	$E_5 \vee E_6 \vee E_{11} \vee E_{12}$	$C_6 \vee C_7$
V_1	$E_1 \vee E_5 \vee E_{12} \vee E_{16}$	$C_1 \vee C_6$
V_2	$E_2 \vee E_6 \vee E_{11} \vee E_{15}$	$C_4 \vee C_7$
ξ_{bc}	$E_1 \vee E_6 \vee E_{11} \vee E_{16}$	$C_1 \vee C_7$
η_{bc}	$E_2 \vee E_5 \vee E_{12} \vee E_{15}$	$C_4 \vee C_6$

Table 2.11: Product events and their equivalence in terms of atomic and correlation events allowed by *Locality Hypothesis*.

$\wedge$	U_1	U_2	V_1	V_2	ξ_{bc}	η_{bc}
U_1	U_1	$\varnothing_{abc}$	C_1	C_4	C_1	C_4
U_2	$\varnothing_{abc}$	U_2	C_6	C_7	C_7	C_6
V_1	C_1	C_6	V_1	$\varnothing_{abc}$	C_1	C_6
V_2	C_4	C_7	$\varnothing_{abc}$	V_2	C_7	C_4
ξ_{bc}	C_1	C_7	C_1	C_7	ξ_{bc}	$\varnothing_{abc}$
η_{bc}	C_4	C_6	C_6	C_4	$\varnothing_{abc}$	η_{bc}

Table 2.12: Events defined on two conditions allowed by *Locality Hypothesis*.

From this table it is now easy to obtain the following relations:

$$i) \quad U_1 \wedge V_1 = U_1 \wedge \xi_{bc} = V_1 \wedge \xi_{bc} = C_1 \qquad (A_a = B_b = B_c) \tag{2.2.46a}$$
$$ii) \quad U_1 \wedge V_2 = U_1 \wedge \eta_{bc} = V_2 \wedge \eta_{bc} = C_4 \qquad (A_a = B_b \neq B_c) \tag{2.2.46b}$$
$$iii) \quad U_2 \wedge V_1 = U_2 \wedge \eta_{bc} = V_1 \wedge \eta_{bc} = C_6 \qquad (A_a = B_c \neq B_b) \tag{2.2.46c}$$
$$iv) \quad U_2 \wedge V_2 = U_2 \wedge \xi_{bc} = V_2 \wedge \xi_{bc} = C_7 \qquad (A_a \neq B_b = B_c) \tag{2.2.46d}$$

These relations are very important because of two reasons:

1. They are a *Locality* and *Realism Criterion* for the boolean algebra of events since they were obtained by applying these hypotheses.

2. They are the connection between the boolean probability algebra and Bell-CHSH inequality given that they introduce *Locality* and *Realism* into the algebraic proof of this last one, as it will be seen later.

In this chapter we have shown that Bohm-EPR experiment supports a local-realistic boolean probability algebra, which has been constructed. The following step is to show the equivalence between this last one and Bell-CHSH inequality. This will be done in the following chapter.

Chapter 3

Algebraic proof of Bell-CHSH inequality

In this chapter we use the boolean probability algebra developed in the previous chapter to give a merely algebraic proof of Bell-CHSH inequality, showing that the essential part of the problem lies on the probability algebra associated to the experiment and not on the hidden-variables.

Let us begin this chapter stating a lemma that will be useful in the algebraic proof of Bell-CHSH inequality.

Lemma 3. *Let T be an arbitrary event and let $\{S_1, S_2\}$ be an arbitrary pair of complementary events. Then:*

$$P(T) = P(T \wedge S_1) + P(T \wedge S_2) \tag{3.0.1}$$

The proof is given in the Appendix.

This lemma is specially handy when used with the pairs of complementary events $\{U_1, U_2\}$, $\{V_1, V_2\}$ and $\{\xi_{bc}, \eta_{bc}\}$.

We now give the algebraic proof of Bell-CHSH inequality. The first step is to state two lemmas, which are *Locality* and *Realism Criteria* for the Correlation Function Q given by (2.1.8), (2.2.23) and (2.2.24).

Lemma 4. *For every choice of parameters a, b and c, the Correlation Function Q for any local-realistic boolean probability algebra satisfies the following inequality:*

$$Q_{ab} - Q_{ac} \leqslant 1 - D_{bc} \tag{3.0.2}$$

where

$$D_{bc} \equiv P(\xi_{bc}) - P(\eta_{bc}) \tag{3.0.3}$$

Proof. From (2.2.23) we have that:

$$Q_{ab} = P(U_1) - P(U_2) \tag{3.0.4}$$

where U_1 and $U_2 \in \mathcal{B}(\Omega_{abc})$.

In a similar way, from (2.2.24) we have that:

$$Q_{ac} = P(V_1) - P(V_2) \tag{3.0.5}$$

where V_1 and $V_2 \in \mathcal{B}(\Omega_{abc})$.

Then:

$$Q_{ab} - Q_{ac} = \Big(P(U_1) - P(U_2)\Big) - \Big(P(V_1) - P(V_2)\Big) \tag{3.0.6}$$

Using *Lemma 3* (3.0.1), with $\{S_1, S_2\} = \{V_1, V_2\}$ in the terms of the first parentheses and $\{S_1, S_2\} = \{U_1, U_2\}$ in the terms of the second parentheses, we get:

$$\begin{aligned} Q_{ab} - Q_{ac} = {} & \Big(P(U_1 \wedge V_1) + P(U_1 \wedge V_2)\Big) - \Big(P(U_2 \wedge V_1) + P(U_2 \wedge V_2)\Big) - \\ & \Big(P(V_1 \wedge U_1) + P(V_1 \wedge U_2)\Big) + \Big(P(V_2 \wedge U_1) + P(V_2 \wedge U_2)\Big) \end{aligned} \tag{3.0.7}$$

Knowing that events are commutative (A.0.1b) and simplifying:

$$Q_{ab} - Q_{ac} = 2\,\Big(P(U_1 \wedge V_2) - P(U_2 \wedge V_1)\Big) \tag{3.0.8}$$

Using the fact that probability functions are always non-negative and that, if $x \geqslant 0$ and $y \geqslant 0$, then $x + y \geqslant x - y$, with $x, y \in \Re^+$, we have that:

$$Q_{ab} - Q_{ac} \leqslant 2\,\Big(P(U_1 \wedge V_2) + P(U_2 \wedge V_1)\Big) \tag{3.0.9}$$

Adding and subtracting $P(U_1 \wedge V_1)$ and $P(U_2 \wedge V_2)$:

$$\begin{aligned} Q_{ab} - Q_{ac} \leqslant {} & 2\,\Big(P(U_1 \wedge V_2) + P(U_2 \wedge V_1)\Big) + \Big(P(U_1 \wedge V_1) - P(U_1 \wedge V_1)\Big) + \\ & \Big(P(U_2 \wedge V_2) - P(U_2 \wedge V_2)\Big) \\ \leqslant {} & \Big(P(U_1 \wedge V_1) + P(U_1 \wedge V_2)\Big) + \Big(P(U_2 \wedge V_1) + P(U_2 \wedge V_2)\Big) + \\ & \Big(P(U_1 \wedge V_2) - P(U_1 \wedge V_1)\Big) + \Big(P(U_2 \wedge V_1) - P(U_2 \wedge V_2)\Big) \end{aligned} \tag{3.0.10}$$

From *Lemma 3* (3.0.1) and rearranging terms we obtain:

$$Q_{ab} - Q_{ac} \leqslant P(U_1) + P(U_2) + \Big(P(U_1 \wedge V_2) + P(U_2 \wedge V_1)\Big) - \Big(P(U_1 \wedge V_1) + P(U_2 \wedge V_2)\Big) \tag{3.0.11}$$

where we have used $\{S_1, S_2\} = \{U_1, U_2\}$.

Knowing that $P(S_1) + P(S_2) = 1$, with $\{S_1, S_2\} = \{U_1, U_2\}$, we get:

$$Q_{ab} - Q_{ac} \leqslant 1 + \Big(P(U_1 \wedge V_2) + P(U_2 \wedge V_1)\Big) - \Big(P(U_1 \wedge V_1) + P(U_2 \wedge V_2)\Big) \quad (3.0.12)$$

Using the relations stated in (2.2.46), we have that:

$$Q_{ab} - Q_{ac} \leqslant 1 + \Big(P(U_1 \wedge \eta_{bc}) + P(U_2 \wedge \eta_{bc})\Big) - \Big(P(U_1 \wedge \xi_{bc}) + P(U_2 \wedge \xi_{bc})\Big) \quad (3.0.13)$$

Finally, using again *Lemma 3* (3.0.1) with $\{S_1, S_2\} = \{U_1, U_2\}$, we obtain:

$$\begin{aligned} Q_{ab} - Q_{ac} &\leqslant 1 + P(\eta_{bc}) - P(\xi_{bc}) \\ &\leqslant 1 - \Big(P(\xi_{bc}) - P(\eta_{bc})\Big) \\ &\leqslant 1 - D_{bc} \ , \end{aligned} \quad (3.0.14)$$

which is the expected result.

□

A remarkable fact is that terms on the left hand side depend on the parameter a, while the terms on the right do not. It means that if we switched parameter a to d, we would get the same result.

From the previous lemma, we can get the following corollary.

Corollary 2. *For every choice of parameters a,b and c, the Correlation Function Q for any local-realistic boolean probability algebra satisfies the following inequality:*

$$\Big|Q_{ab} - Q_{ac}\Big| \leqslant 1 - D_{bc} \quad (3.0.15)$$

Proof. From the equations (3.0.2) and (3.0.3) we have that:

$$Q_{ab} - Q_{ac} \leqslant 1 - D_{bc} \quad (3.0.16)$$

On the other hand, from *Corollary 1* (2.2.29), we have that $D_{bc} = D_{cb}$. Hence:

$$\begin{aligned} Q_{ac} - Q_{ab} &\leqslant 1 - D_{cb} \\ &\leqslant 1 - D_{bc} \end{aligned} \quad (3.0.17)$$

Now, let $x = Q_{ab} - Q_{ac}$ and $y = 1 - D_{bc}$ be. Then, knowing that $1 - D_{bc} \geqslant 0$, we obtain, from (3.0.16) and (3.0.17): $x \leqslant y$ and $-x \leqslant y$. This can be expressed in a different way: $|x| \leqslant y$.

Thus:

$$\left| Q_{ab} - Q_{ac} \right| \leqslant 1 - D_{bc} \ , \tag{3.0.18}$$

which is the expected result.

□

We now state the second lemma.

Lemma 5. *For every choice of parameters a, b and c, the Correlation Function Q for any local-realistic boolean probability algebra satisfies the following inequality:*

$$Q_{ab} + Q_{ac} \leqslant 1 + D_{bc} \tag{3.0.19}$$

Proof. From (2.2.23) we have that:

$$Q_{ab} = P(U_1) - P(U_2) \tag{3.0.20}$$

where U_1 and $U_2 \in \mathcal{B}(\Omega_{abc})$.

In a similar way, from (2.2.24) we have that:

$$Q_{ac} = P(V_1) - P(V_2) \tag{3.0.21}$$

where V_1 and $V_2 \in \mathcal{B}(\Omega_{abc})$.

Then:

$$Q_{ab} + Q_{ac} = \Big(P(U_1) - P(U_2)\Big) + \Big(P(V_1) - P(V_2)\Big) \tag{3.0.22}$$

Using *Lemma 3* (3.0.1), with $\{S_1, S_2\} = \{V_1, V_2\}$ in the terms of the first parentheses and $\{S_1, S_2\} = \{U_1, U_2\}$ in the terms of the second parentheses, we get:

$$\begin{aligned} Q_{ab} + Q_{ac} = &\Big(P(U_1 \wedge V_1) + P(U_1 \wedge V_2)\Big) - \Big(P(U_2 \wedge V_1) + P(U_2 \wedge V_2)\Big) + \\ &\Big(P(V_1 \wedge U_1) + P(V_1 \wedge U_2)\Big) - \Big(P(V_2 \wedge U_1) + P(V_2 \wedge U_2)\Big) \end{aligned} \tag{3.0.23}$$

Knowing that events are commutative (A.0.1b) and simplifying:

$$Q_{ab} + Q_{ac} = 2 \Big(P(U_1 \wedge V_1) - P(U_2 \wedge V_2)\Big) \tag{3.0.24}$$

Using the fact that probability functions are always non-negative and that, if $x \geqslant 0$ and $y \geqslant 0$, then $x + y \geqslant x - y$, with $x, y \in \Re^+$, we have that:

$$Q_{ab} + Q_{ac} \leqslant 2 \Big(P(U_1 \wedge V_1) + P(U_2 \wedge V_2)\Big) \tag{3.0.25}$$

Adding and subtracting $P(U_1 \wedge V_2)$ and $P(U_2 \wedge V_1)$:

$$\begin{aligned}
Q_{ab} + Q_{ac} &\leqslant 2 \Big(P(U_1 \wedge V_1) + P(U_2 \wedge V_2)\Big) + \Big(P(U_1 \wedge V_2) - P(U_1 \wedge V_2)\Big) + \\
&\quad \Big(P(U_2 \wedge V_1) - P(U_2 \wedge V_1)\Big) \\
&\leqslant \Big(P(U_1 \wedge V_1) + P(U_1 \wedge V_2)\Big) + \Big(P(U_2 \wedge V_1) + P(U_2 \wedge V_2)\Big) + \\
&\quad \Big(P(U_1 \wedge V_1) - P(U_1 \wedge V_2)\Big) + \Big(P(U_2 \wedge V_2) - P(U_2 \wedge V_1)\Big)
\end{aligned} \tag{3.0.26}$$

From *Lemma 3* (3.0.1) and rearranging terms we obtain:

$$Q_{ab} + Q_{ac} \leqslant P(U_1) + P(U_2) + \Big(P(U_1 \wedge V_1) + P(U_2 \wedge V_2)\Big) - \Big(P(U_1 \wedge V_2) + P(U_2 \wedge V_1)\Big) , \tag{3.0.27}$$

where we have used $\{S_1, S_2\} = \{U_1, U_2\}$.

Knowing that $P(S_1) + P(S_2) = 1$, with $\{S_1, S_2\} = \{U_1, U_2\}$, get:

$$Q_{ab} + Q_{ac} \leqslant 1 + \Big(P(U_1 \wedge V_1) + P(U_2 \wedge V_2)\Big) - \Big(P(U_1 \wedge V_2) + P(U_2 \wedge V_1)\Big) \tag{3.0.28}$$

Using the relations stated in (2.2.46), we have that:

$$Q_{ab} + Q_{ac} \leqslant 1 + \Big(P(U_1 \wedge \xi_{bc}) + P(U_2 \wedge \xi_{bc})\Big) - \Big(P(U_1 \wedge \eta_{bc}) + P(U_2 \wedge \eta_{bc})\Big) \tag{3.0.29}$$

Finally, using again *Lemma 3* (3.0.1) with $\{S_1, S_2\} = \{U_1, U_2\}$, we obtain:

$$\begin{aligned}
Q_{ab} + Q_{ac} &\leqslant 1 + P(\xi_{bc}) - P(\eta_{bc}) \\
&\leqslant 1 + \Big(P(\xi_{bc}) - P(\eta_{bc})\Big) \\
&\leqslant 1 + D_{bc} \ ,
\end{aligned} \tag{3.0.30}$$

which is the expected result.

□

As before, terms on the left hand side depends on parameter a, while the terms to the right do not. Switching parameter a to d produces the same result.

Based on *Corollary 2* (3.0.15) and *Lemma 5* (3.0.19), we now state the Bell-CHSH inequality.

Theorem 4. *For every choice of parameters a, b ,c and d, the Correlation Function Q for any local-realistic boolean probability algebra satisfies the following inequality:*

$$\Big|Q_{ab} - Q_{ac}\Big| + Q_{db} + Q_{dc} \leqslant 2 \tag{3.0.31}$$

Proof. From *Corollary 2* (3.0.15) we have that:

$$\left| Q_{ab} - Q_{ac} \right| \leqslant 1 - D_{bc} \tag{3.0.32}$$

On the other hand, from *Lemma 5* (3.0.19):

$$Q_{db} + Q_{dc} \leqslant 1 + D_{bc} \tag{3.0.33}$$

where we have switched a to d.

Adding (3.0.32) and (3.0.33), we obtain:

$$\left| Q_{ab} - Q_{ac} \right| + Q_{db} + Q_{dc} \leqslant 2 \tag{3.0.34}$$

which is the expected result.

□

We have thereby shown that Correlation Function Q satisfies Bell-CHSH inequality in addition to two new *Locality* and *Realism Criteria*. Then, because the proof is merely algebraic, boolean probability algebra over which Q is defined and Bell-CHSH inequality are equivalent.

In the following chapter we will show that any hidden-variable physical model corresponds to a local-realistic boolean probability algebra, evincing that both approaches are completely equivalent.

Chapter 4

Equivalence between hidden-variables models and boolean probability algebras

The aim of this chapter is to show that probabilities over any hidden-variable physical model correspond to those ones for a local-realistic boolean probability algebra.

4.1 The Γ-phase space

4.1.1 Regions of Γ

In Chapter 1, we said that Bell introduced the Γ-phase space as a way of representing the space where λ lies in. Let us now study its properties carefully.

According to *Realism Hypothesis*, each region of Γ contains the information for all the possible configurations of the experiment. On the other hand, *Locality Hypothesis* states that the asymptotic value of λ keeps a unique value for A_a, B_b and B_c. Therefore, regions with a distinct value for A_a depending on the configuration are prohibited.

Then, the local-realistic region of Γ can be divided in 8 regions, named ***origin regions***, which are shown below:

$$\Gamma_1 = \{\lambda | A_a(\lambda) = +1,\ B_b(\lambda) = +1,\ B_c(\lambda) = +1\} \tag{4.1.1a}$$
$$\Gamma_2 = \{\lambda | A_a(\lambda) = +1,\ B_b(\lambda) = +1,\ B_c(\lambda) = -1\} \tag{4.1.1b}$$
$$\Gamma_3 = \{\lambda | A_a(\lambda) = +1,\ B_b(\lambda) = -1,\ B_c(\lambda) = +1\} \tag{4.1.1c}$$
$$\Gamma_4 = \{\lambda | A_a(\lambda) = +1,\ B_b(\lambda) = -1,\ B_c(\lambda) = -1\} \tag{4.1.1d}$$
$$\Gamma_5 = \{\lambda | A_a(\lambda) = -1,\ B_b(\lambda) = +1,\ B_c(\lambda) = +1\} \tag{4.1.1e}$$
$$\Gamma_6 = \{\lambda | A_a(\lambda) = -1,\ B_b(\lambda) = +1,\ B_c(\lambda) = -1\} \tag{4.1.1f}$$
$$\Gamma_7 = \{\lambda | A_a(\lambda) = -1,\ B_b(\lambda) = -1,\ B_c(\lambda) = +1\} \tag{4.1.1g}$$
$$\Gamma_8 = \{\lambda | A_a(\lambda) = -1,\ B_b(\lambda) = -1,\ B_c(\lambda) = -1\} \tag{4.1.1h}$$

Thus, when we carry out a measurement, variables A_a, B_b and B_c take the value specified in the brackets according to the λ determined at the time of production process.

As origin regions are sets of values of λ, from now on we will treat them as sets.

Using the Set Theory and the union ($\cup$) and intersection ($\cap$) operations, we can express the characteristics of the origin regions in a more adequate way:

$$i) \qquad \Gamma_1 \cup \cdots \cup \Gamma_8 = \Gamma \tag{4.1.2a}$$

$$ii) \qquad \Gamma_i \cap \Gamma_j = \begin{cases} \Gamma_i & \text{if } i = j \text{ , with } i, j = 1, \ldots, 8 \\ \emptyset & \text{if } i \neq j \text{ , with } i, j = 1, \ldots, 8 \end{cases} \tag{4.1.2b}$$

where Γ is the local-realistic region of the phase space and $\emptyset$ is the empty region.

In addition, it is possible to construct *any* region of Γ from the eight origin regions (4.1.1). Let γ be an arbitrary region (distinct of $\emptyset$) of Γ. Then:

$$\gamma = \Gamma_i \cup \cdots \cup \Gamma_m \ , \tag{4.1.3}$$

where $i, \ldots, m = 1, \ldots, 8$.

4.1.2 The Probability Integral over Γ

So far, we have just discussed the origin regions of Γ and their properties, but we have not said anything about the integral $\int d\rho$ yet. This integral is a function that takes a region from Γ and returns a numerical value between 0 and 1. In particular, it complies with the following:

$$i) \qquad \int_\Gamma d\rho \ = 1 \tag{4.1.4a}$$

$$ii) \qquad \int_\emptyset d\rho \ = 0 \tag{4.1.4b}$$

Also, from Set Theory, we have that if γ_1 and γ_2 are two arbitrary (not necessarily origin) regions , it turns out that:

$$\int_{\gamma_1 \cup \gamma_2} d\rho \ = \int_{\gamma_1} d\rho \ + \int_{\gamma_2} d\rho \ - \int_{\gamma_1 \cap \gamma_2} d\rho \ , \tag{4.1.5}$$

In particular, for the case where γ_1 and γ_2 are two different origin regions, say Γ_k and Γ_l, with $k \neq l$, we get, using (4.1.2b) and (4.1.4b):

$$\int_{\Gamma_k \cup \Gamma_l} d\rho \ = \int_{\Gamma_k} d\rho \ + \int_{\Gamma_l} d\rho \tag{4.1.6}$$

where k, $l = 1, \ldots, 8$.

So, as any arbitrary region γ (distinct from $\emptyset$) can be expressed as the union of origin regions (4.1.3), it turns out that:

$$\int_\gamma d\rho \ = \int_{\Gamma_i \cup \cdots \cup \Gamma_m} d\rho = \int_{\Gamma_i} d\rho \ + \ \cdots \ + \int_{\Gamma_m} d\rho \tag{4.1.7}$$

where $i, \ldots, m = 1, \ldots, 8$.

The usefulness of this equation lies on that all can be expressed in terms of the origin regions (4.1.1).

4.2 Connection of Γ with the boolean probability algebras

In the previous subsection, we emphasized that the Probability Integral is a function over Γ which returns a numerical value between 0 and 1. Let us now denote this function as $\wp$. With this notation, properties of the Probability Integral over Γ can be rewritten in the following way:

$$\int_{\Gamma} d\rho \ = 1 \quad \longrightarrow \quad \wp(\Gamma) = 1 \tag{4.2.1a}$$

$$\int_{\emptyset} d\rho \ = 0 \quad \longrightarrow \quad \wp(\emptyset) = 0 \tag{4.2.1b}$$

$$\int_{\gamma_1 \cup \gamma_2} d\rho \ = \int_{\gamma_1} d\rho \ + \int_{\gamma_2} d\rho \ - \int_{\gamma_1 \cap \gamma_2} d\rho \quad \longrightarrow \quad \wp(\gamma_1 \cup \gamma_2) = \wp(\gamma_1) + \wp(\gamma_2) - \wp(\gamma_1 \cap \gamma_2) \tag{4.2.1c}$$

with γ_1 and γ_2 arbitrary regions of Γ.

In particular, given that from (4.1.3) an arbitrary region γ can be expressed in terms of the origin regions (4.1.1), we have that:

$$\int_{\gamma} d\rho \ = \int_{\Gamma_i} d\rho \ + \ \cdots \ + \int_{\Gamma_m} d\rho \quad \longrightarrow \quad \wp(\gamma) = \wp(\Gamma_i) + \cdots + \wp(\Gamma_m) \tag{4.2.2}$$

where $\Gamma_i, \ldots, \Gamma_m$ are origin regions.

With this notation we can realize that the function acting over Γ does not have to be an integral but *any* function such that satisfies the properties stated above. Then, it is evident that $\wp$ corresponds to the Probability Distribution Function P.

The fact that we can associate the integral $\int d\rho$ with the Probability Distribution Function P gives us the "sight" of a boolean probability algebra. To see it clearer, let us denote the operations $\cup$ and $\cap$ as $\oplus$ and $\otimes$, respectively. In this way:

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_8 \quad \longrightarrow \quad \Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_8 \tag{4.2.3a}$$

$$\Gamma_i \cap \Gamma_j = \begin{cases} \Gamma_i & \text{if } i = j \\ \emptyset & \text{if } i \neq j \end{cases} \quad \longrightarrow \quad \Gamma_i \otimes \Gamma_j = \begin{cases} \Gamma_i & \text{if } i = j \\ \emptyset & \text{if } i \neq j \end{cases} \tag{4.2.3b}$$

$$\gamma = \Gamma_i \cup \cdots \cup \Gamma_m \quad \longrightarrow \quad \gamma = \Gamma_i \oplus \cdots \oplus \Gamma_m \tag{4.2.3c}$$

where $i, m = 1, \ldots, 8$.

Then, it is easy to realize that the origin regions do not have to be "necessarily" sets but elements of any boolean algebra, in such a way that they could even not make reference to sets of λ, but *the event itself* of A_a, B_b and B_c obtaining a specific value.

This fact has a great transcendence because we are going from an algebra of sets to a boolean algebra of events. The main difference is that while in the first one everything turns around λ (i.e., λ determines the entire system), in the second one the essential part are the events *themselves*, without the necessity of recurring to the hidden-variable λ.

Given that any region of Γ can be gotten from its eight origin regions (4.1.1) and that under locality any event of $\mathcal{B}(\Omega_{abc})$ can be obtained from its eight simple events (Table 2.9), it is enough to get the connection between the origin regions and the simple events and between the operations $\int d\rho$ and P to show the equivalence between hidden-variable models and local-realistic boolean algebras of events.

Definition 7. *The connection of the origin regions of* Γ (4.1.1) *and the simple events of* $\mathcal{B}(\Omega_{abc})$ *under locality (Table 2.9) is given by the following relations:*

$$\Gamma_1 \equiv E_1 \tag{4.2.4a}$$
$$\Gamma_2 \equiv E_2 \tag{4.2.4b}$$
$$\Gamma_3 \equiv E_5 \tag{4.2.4c}$$
$$\Gamma_4 \equiv E_6 \tag{4.2.4d}$$
$$\Gamma_5 \equiv E_{11} \tag{4.2.4e}$$
$$\Gamma_6 \equiv E_{12} \tag{4.2.4f}$$
$$\Gamma_7 \equiv E_{15} \tag{4.2.4g}$$
$$\Gamma_8 \equiv E_{16} \tag{4.2.4h}$$

Definition 8. *The connection between the Probability Integral over* Γ *and the Probability Function over* $\mathcal{B}(\Omega_{abc})$ *is given by the following relations:*

$$\int_{\Gamma_1} d\rho \equiv P(E_1) \tag{4.2.5a}$$
$$\int_{\Gamma_2} d\rho \equiv P(E_2) \tag{4.2.5b}$$
$$\int_{\Gamma_3} d\rho \equiv P(E_5) \tag{4.2.5c}$$
$$\int_{\Gamma_4} d\rho \equiv P(E_6) \tag{4.2.5d}$$
$$\int_{\Gamma_5} d\rho \equiv P(E_{11}) \tag{4.2.5e}$$
$$\int_{\Gamma_6} d\rho \equiv P(E_{12}) \tag{4.2.5f}$$
$$\int_{\Gamma_7} d\rho \equiv P(E_{15}) \tag{4.2.5g}$$
$$\int_{\Gamma_8} d\rho \equiv P(E_{16}) \tag{4.2.5h}$$

We will now use these relations to show the equivalence of the definitions of Q and D in both approaches.

Lemma 6. ***Correlation Function:*** *For every choice of parameters a, b and c, the Correlation Functions Q and D satisfy the following equivalences:*

$$Q_{ab} = \int_{\Gamma} A_a(\lambda)\, B_b(\lambda)\, d\rho \;=\; P(U_1) - P(U_2) \tag{4.2.6a}$$

$$Q_{ac} = \int_{\Gamma} A_a(\lambda)\, B_c(\lambda)\, d\rho \;=\; P(V_1) - P(V_2) \tag{4.2.6b}$$

$$D_{bc} = \int_{\Gamma} B_b(\lambda)\, B_c(\lambda)\, d\rho \;=\; P(\xi_{bc}) - P(\eta_{bc}) \tag{4.2.6c}$$

We give the proof for the first expression. The other two have similar demonstrations.

Proof. From *Definition 2* (1.2.1) we have that:

$$Q_{ab} = \int_{\Gamma} A_a(\lambda)\, B_b(\lambda)\, d\rho \tag{4.2.7}$$

Using (4.1.2a) and (4.1.6) we obtain:

$$\begin{aligned} Q_{ab} &= \sum_{i=1}^{8} \int_{\Gamma_i} A_a(\lambda) B_b(\lambda) \rho(\lambda) d\lambda \\ &= \int_{\Gamma_1} d\rho \,+ \int_{\Gamma_2} d\rho \,- \int_{\Gamma_3} d\rho \,- \int_{\Gamma_4} d\rho \,- \int_{\Gamma_5} d\rho \,- \int_{\Gamma_6} d\rho \,+ \int_{\Gamma_7} d\rho \,+ \int_{\Gamma_8} d\rho \end{aligned} \tag{4.2.8}$$

where we have used the values of A_a and B_b for the eight origin regions (4.1.1). Using the equivalences from *Definition 8* (4.2.5) and rearranging terms:

$$\begin{aligned} Q_{ab} =& \Big(P(E_1) + P(E_2) + P(E_{15}) + P(E_{16})\Big) - \\ & \Big(P(E_5) + P(E_6) + P(E_{11}) + P(E_{12})\Big) \end{aligned} \tag{4.2.9}$$

Using the point 2b of *Lemma 7* (Appendix C) and the values from Table 2.11 we obtain:

$$\begin{aligned} Q_{ab} &= P\Big(E_1 \vee E_2 \vee E_{15} \vee E_{16}\Big) - P\Big(E_5 \vee E_6 \vee E_{11} \vee E_{12}\Big) \\ &= P(U_1) - P(U_2) \end{aligned} \tag{4.2.10}$$

which is the expected result.

□

Since the Correlation Function Q is the starting point in both approaches, we have thus shown the equivalence between hidden-variable models and boolean probability algebras.

Then, any hidden-variable physical model supported by the experiment, regardless of production process and its physical realization, admits a local-realistic boolean probability algebra, being this last one the essential part of the problem.

Conclusions

This thesis shows that the following statements are equivalent:

1. There is a hidden-variable model based on local Realism for the experiment.

2. The experiment supports a local-realistic boolean probability algebra.

3. Correlation Function for the experiment satisfies Bell-CHSH inequalities.

The equivalence was obtained by algebraic methods. In particular, a merely algebraic proof of Bell-CHSH inequality was given.

Appendix A

General properties of the boolean algebras [8]

A boolean algebra $\mathfrak{A}$ is a set of elements X, Y, Z, ... endowed of two binary operations called *sum* and *product*, denoted respectively as $\vee$ and $\wedge$, and a monary operation called *complement*, denoted with a prime ($'$), with the following properties:

1. **Commutativity:**

$$X \vee Y = Y \vee X \tag{A.0.1a}$$
$$X \wedge Y = Y \wedge X \tag{A.0.1b}$$

2. **Associativity:**

$$(X \vee Y) \vee Z = X \vee (Y \vee Z) \tag{A.0.2a}$$
$$(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z) \tag{A.0.2b}$$

3. **Distributivity:**

$$(X \vee Y) \wedge Z = (X \wedge Z) \vee (Y \wedge Z) \tag{A.0.3a}$$
$$(X \wedge Y) \vee Z = (X \vee Z) \wedge (Y \vee Z) \tag{A.0.3b}$$

4. **Neutral elements:**

There is a neutral element $\varnothing$ for the sum such that $X \vee \varnothing = X$, $\forall X \in \mathfrak{A}$. (A.0.4a)

There is a neutral element $\mathbb{I}$ for the product such that $X \wedge \mathbb{I} = X$, $\forall X \in \mathfrak{A}$. (A.0.4b)

5. **Complement:** $\forall X \in \mathfrak{A}$ there is an $X' \in \mathfrak{A}$ called ***complement*** of X, such that:

$$X \vee X' = \mathbb{I} \tag{A.0.5a}$$
$$X \wedge X' = \varnothing \tag{A.0.5b}$$

From here, we can obtain other important properties:

1. **Idempotency:**

$$X \vee X = X \tag{A.0.6a}$$
$$X \wedge X = X \tag{A.0.6b}$$

2. **Maximality of $\mathbb{I}$ and minimality of $\varnothing$:**

$$X \vee \mathbb{I} = \mathbb{I} \tag{A.0.7a}$$
$$X \wedge \varnothing = \varnothing \tag{A.0.7b}$$

3. **Involution:**

$$(X')' = X \tag{A.0.8a}$$

4. **Immersion:**

$$X \vee (X \wedge Y) = X \tag{A.0.9a}$$
$$X \wedge (X \vee Y) = X \tag{A.0.9b}$$

5. **Morgan's Law:**

$$(X \vee Y)' = X' \wedge Y' \tag{A.0.10a}$$
$$(X \wedge Y)' = X' \vee Y' \tag{A.0.10b}$$

Appendix B

Boolean algebras of events

We will start by giving two important definitions.

Definition 9. ***Sample space*** $(\mathcal{S})$**:** *is the collection of all possible outcomes of an experiment or trial.*

Definition 10. ***Event:*** *is any collection of possible outcomes of an experiment or trial.*

Let us now consider an experiment that consists on N different outcomes. Then, from these ones it is possible to form a boolean algebra of events of order N, denoted as $\mathcal{B}(S)$, which is a set which includes every collection of events from the sample space S and is closed under the operations *conjunction* $\wedge$ (AND) and *disjunction* $\vee$ (OR).

Such an algebra consists on 2^N events, which have the following classification:

1. **Impossible event** $(\varnothing)$**:** is an event that never occurs.
2. **Simple or atomic event** (E)**:** is any event that consists of exactly a ***single*** outcome from the experiment.
3. **Compound event:** is any event that consists of more than one outcome from the experiment.
4. **Complementary event:** consists of all the outcomes ***not*** in the original event. The complementary of $X \in \mathcal{B}(\mathcal{S})$ is denoted by X' $(X' \in \mathcal{B}(\mathcal{S}))$.
5. **Certain event** $(\mathbb{I})$**:** is an event that always occurs.

In particular, simple events have the following properties:

$$i) \qquad E_1 \vee \cdots \vee E_N = \mathbb{I} \tag{B.0.1a}$$

$$ii) \qquad E_i \wedge E_j = \begin{cases} E_i & \text{if } i = j, \text{ with } i, j = 1, \ldots, N \\ \varnothing & \text{if } i \neq j, \text{ with } i, j = 1, \ldots, N \end{cases} \tag{B.0.1b}$$

where $\mathbb{I}$ and $\varnothing$ are the ***certain*** and ***impossible events***, respectively, of $\mathcal{B}(\mathcal{S})$.

Appendix C

Probability Function over a boolean algebra of events

Definition 11. *An ordered pair* $< \mathcal{B}(\mathcal{S}), P >$ *is a* ***probability algebra*** *if* $\mathcal{B}(\mathcal{S})$ *is a boolean algebra of events and* P *a real-value function (a* ***Probability Function****) defined on elements of the universe of* $\mathcal{B}(\mathcal{S})$ *which is:*

1. ***Strictly positive:*** $P(X) \geqslant 0, \ \forall X \in \mathcal{B}(\mathcal{S})$.
   Moreover: $P(X) = 0$ *if and only if* $X = \varnothing$.
2. ***Normed:*** $P(X) = 1$ *if and only if* $X = \mathbb{I}$.
3. ***Additive:*** $P(X \vee Y) = P(X) + P(Y) - P(X \wedge Y), \forall X, Y \in \mathcal{B}(\mathcal{S})$.

From this definition it is possible to obtain the following properties for any probability algebra.

Lemma 7. ***Properties of a probability algebra***

1. $P(X) + P(X') = 1, \forall X \in \mathcal{B}(\mathcal{S})$.
2. *Let* E_i *and* $E_j \in \mathcal{B}(\mathcal{S})$, *with* $i \neq j$, *two different atomic events. Then:*
   (a) $P(E_i \wedge E_j) = 0$ *because* $E_i \wedge E_j = \varnothing$.
   (b) $P(E_i \vee E_j) = P(E_i) + P(E_j)$.
3. *Let* $E_1, \ldots, E_N$ *the* N *atomic events of* $\mathcal{B}(\mathcal{S})$. *Then:* $\sum_{i=1}^{N} P(E_i) = 1$.

Elements of a probability algebra are referred to as *events*.

Let us now state a handy lemma to be used with pairs of complementary events.

Lemma 8. *Let* T *be an arbitrary event and let* $\{S_1, S_2\}$ *be an arbitrary pair of complementary events. Then:*

$$P(T) = P(T \wedge S_1) + P(T \wedge S_2) \tag{C.0.1}$$

Proof. First, let us prove two important relations.

$$
\begin{aligned}
i) \qquad T &= T \wedge \mathbb{I} \\
&= T \wedge (S_1 \vee S_2) \\
&= (T \wedge S_1) \vee (T \wedge S_2)
\end{aligned}
\tag{C.0.2}
$$

$$
\begin{aligned}
ii) \qquad & T \wedge (S_1 \wedge S_2 = \varnothing) \\
& (T \wedge T) \wedge (S_1 \wedge S_2) = T \wedge \varnothing \\
& (T \wedge S_1) \wedge (T \wedge S_2) = \varnothing
\end{aligned}
\tag{C.0.3}
$$

Using these relations we get:

$$
\begin{aligned}
P(T) &= P\Big((T \wedge S_1) \vee (T \wedge S_2)\Big) \\
&= P(T \wedge S_1) + P(T \wedge S_2) - P\Big((T \wedge S_1) \wedge (T \wedge S_2)\Big) \\
&= P(T \wedge S_1) + P(T \wedge S_2)
\end{aligned}
\tag{C.0.4}
$$

where we have used the additive property of the Probability Function (point 3 of *Definition 11*) and that $P(\varnothing) = 0$.

□

Appendix D

Classification of events from $\mathcal{B}(\Omega_{ab})$

Events from $\mathcal{B}(\Omega_{ab})$ are classified in the following way:

1. **Impossible event:**

$$\varnothing_{ab} \tag{D.0.1a}$$

2. **Simple or atomic events:**

$$(+1,+1)_{ab} \tag{D.0.2a}$$
$$(+1,-1)_{ab} \tag{D.0.2b}$$
$$(-1,+1)_{ab} \tag{D.0.2c}$$
$$(-1,-1)_{ab} \tag{D.0.2d}$$

3. **Compound events:**

$$(+1,\, \mathbb{I}_b) \equiv \{A_a = +1\} = (+1,+1)_{ab} \vee (+1,-1)_{ab} \tag{D.0.3a}$$
$$(-1,\, \mathbb{I}_b) \equiv \{A_a = -1\} = (-1,+1)_{ab} \vee (-1,-1)_{ab} \tag{D.0.3b}$$
$$(\mathbb{I}_a,\, +1) \equiv \{B_b = +1\} = (+1,+1)_{ab} \vee (-1,+1)_{ab} \tag{D.0.3c}$$
$$(\mathbb{I}_a,\, -1) \equiv \{B_b = -1\} = (+1,-1)_{ab} \vee (-1,-1)_{ab} \tag{D.0.3d}$$
$$\varepsilon_{ab} \equiv \{A_a = B_b\} = (+1,+1)_{ab} \vee (-1,-1)_{ab} \tag{D.0.3e}$$
$$\delta_{ab} \equiv \{A_a = -B_b\} = (+1,-1)_{ab} \vee (-1,+1)_{ab} \tag{D.0.3f}$$

4. **Complementary events:**

$$(+1,+1)'_{ab} = (+1,-1)_{ab} \vee (-1,+1)_{ab} \vee (-1,-1)_{ab} \tag{D.0.4a}$$
$$(+1,-1)'_{ab} = (+1,+1)_{ab} \vee (-1,+1)_{ab} \vee (-1,-1)_{ab} \tag{D.0.4b}$$
$$(-1,+1)'_{ab} = (+1,+1)_{ab} \vee (+1,-1)_{ab} \vee (-1,-1)_{ab} \tag{D.0.4c}$$
$$(-1,-1)'_{ab} = (+1,+1)_{ab} \vee (+1,-1)_{ab} \vee (-1,+1)_{ab} \tag{D.0.4d}$$

5. **Certain event:**

$$\mathbb{I}_{ab} = (+1,+1)_{ab} \vee (+1,-1)_{ab} \vee (-1,+1)_{ab} \vee (-1,-1)_{ab} \tag{D.0.5a}$$

The *Hasse Diagram* for this boolean algebra of events is shown in Figure D. In this scheme it is possible to see in a graphical way the relation among the events stated above.

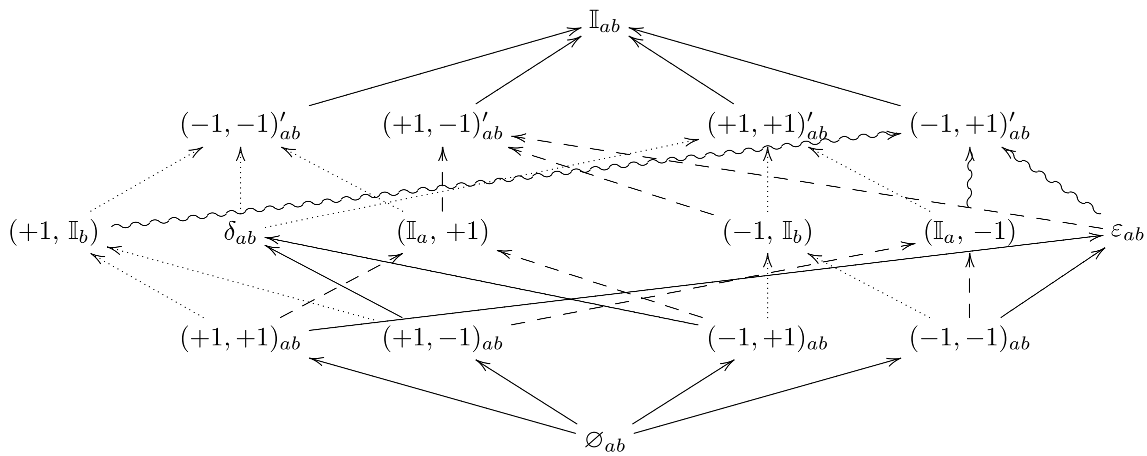

Figure D: Hasse Diagram for the boolean algebra of events of the Bohm-EPR experiment with one measurement parameter per site.

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