UNIVERSIDAD AUTÓNOMA DE SAN LUIS POTOSÍ

FACULTAD DE CIENCIAS

"TWO PROBLEMS ABOUT CONVEX POLYGONS IN DISCRETE GEOMETRY"

TESIS

PARA OBTENER EL GRADO DE

DOCTOR EN CIENCIAS APLICADAS

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Yo, Mario Lomelí Haro, estudiante del Posgrado en Ciencias Aplicadas de la Facultad de Ciencias de la Universidad Autónoma de San Luis Potosí, como autor de la tesis "*Two problems about convex polygons in Discrete Geometry*",declaro que la tesis es una obra original, inédita, auténtica, personal, que se han citado las fuentes correspondientes y que en su ejecución se respetaron las disposiciones legales vigentes que protegen los derechos de autor y de propiedad intelectual e industrial. Las ideas, doctrinas, resultados y conclusiones a los que he llegado son de mi absoluta responsabilidad.

I, Mario Lomelí Haro, student of the graduate program Posgrado en Ciencias Aplicadas of the Facultad de Ciencias in the Universidad Autónoma de San Luis Potosí and as the author of the thesis entitled "*Two problems about convex polygons in Discrete Geometry*", hereby declare that this thesis is an original, undistributed, authentic and personal work for which the corresponding sources have been cited appropriately and that in their execution the legal provisions were respected protecting copyright, intellectual and industrial property. The ideas, doctrines, results and conclusions that I have reached are of my absolute responsibility.

Agradecimientos

A mi familia.

A los doctores Jesús Leaños, Gelasio Salazar y Jorge Urrutia, que me tuvieron tanta paciencia y por todo lo que me compartieron.

Al CONACyT, por las becas otorgadas al becario número 163435.

A Dios, por todo lo que me ha dado, que siempre han sido cosas buenas.

Dedicatoria

A mi familia, a mis seres queridos, todas esas personas que han estado conmigo.

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is right,
Because their words had forked no lightning they
Do not go gentle into that good night.

Rage, rage against the dying of the light.

Dylan Thomas.
Fragmento recitado en Interstellar.

Resumen

En este trabajo abordamos dos problemas sobre polígonos convexos, con vértices en un conjunto finito de puntos en el plano, en posición general.

En el primer problema, obtenemos el dibujo rectilíneo de la gráfica completa, con vértices en la colección de puntos dada, y colorearemos las aristas de acuerdo a la siguiente regla: para cualesquiera dos aristas, si sus cerraduras son disjuntas, entonces deben tener diferente color. Estudiaremos el número de colores necesarios y suficientes que debe tener cualquier coloración de este tipo.

En este trabajo, encontramos el número de colores necesarios y suficientes para colorear las aristas de la gráfica completa, cuando los vértices están en la doble cadena. Cabe mencionar que, este problema estaba resuelto únicamente para cuando la colección de puntos es el conjunto de vértices de un polígono convexo.

Probaremos que si coloreamos, de manera óptima, las aristas con extremos en la cadena mayor, y posteriormente, seleccionando cada vértice restante, y coloreando del mismo color sus aristas incidentes, entonces el número de colores empleados es el óptimo.

De nuestro resultado, conjeturamos que, para obtener una coloración óptima en cualquier colección de puntos, se procede de manera semejante: colorear, de manera óptima, las aristas con extremos en el polígono convexo más grande, y posteriormente, seleccionando cada vértice para colorear de un solo color todas las aristas incidentes a él.

Formalmente, estamos coloreando, de manera óptima, la gráfica de disjuntez $G_D = (V_D, E_D)$, inducida de un conjunto P de puntos en el plano en posición general. Obtenemos a G_D de la siguiente manera: tomamos el dibujo rectilíneo $\mathcal{D}$ de $K_{|P|} = (P, E)$ y hacemos $V_D = E$, y $E_D = \{ee' :$ las cerraduras de e y de e' no se intersectan en $\mathcal{D}\}$. Que, como mencionamos, tal número cromático sólo es conocido para cuando P es el conjunto de vértices de un polígono convexo. Probamos que la gráfica de disjuntez, en la doble cadena, tiene un número cromático sustancialmente más grande que el que se conoce.

En el segundo problema que estudiamos, nos interesa que los polígonos sean vacíos, y no necesariamente buscamos el más grande. Buscamos particionar el cierre convexo del conjunto de puntos dado P con polígonos con-

vexos, con interiores disjuntos, cuyos vértices estén en P. A este conjunto de polígonos se le llama descomposición convexa. Estamos interesados en descomposiciones convexas de cardinalidad mínima.

Por poner un ejemplo, tenemos a las triangulaciones: descomposiciones convexas cuyos elementos, desde luego, son triángulos. El número de elementos en una triangulación es bien conocido. Si pudiéramos encontrar una descomposición convexa cuyos elementos fueran cuadriláteros, su cardinalidad sería la de una triangulación dividida por dos. Pero es fácil encontrar colecciones de puntos que no admiten cuadrilaterizaciones, en las que todos sus elementos sean convexos. Partiremos de una triangulación específica y encontraremos aristas que pueden ser eliminadas para obtener polígonos convexos más grandes, reduciendo la cardinalidad de la descomposición.

Preface

In this work we study two problems involving convex polygons with its vertices in a given point set. In the first problem, we obtain a rectilinear drawing of the complete graph and color the edges according to the unique rule: for any given two edges, if their closures are disjoint, then they must have different color. We analyze how to color all the edges using as fewest colors as possible, when the set of vertices is the double chain. We prove that to use the fewest colors, first, we color the edges with its endpoints in the biggest convex chain in an optimal way, and second, we select every remaining point and color all of its incident edges with a new different color.

We conjecture that the optimal way of coloring the edges of the complete graph with vertices in any given point set is similar: first obtain the biggest subset of points in convex position (without caring if it is empty or has another elements in its interior) to color all its incident edges in an optimal way, and second, color the remaining edges by coloring with one different color all the edges incident with each of the remaining vertices.

Formally, we are studying the *chromatic number of the edge disjointness graph* of the complete graph, with its vertex set in a given point set. As far as we know, such chromatic number is only known when the point set is the set of vertices of a convex polygon. In the double chain we obtain a number substantially bigger than the previous one.

In the second problem, we care if the polygons are empty, and we may not be interested in the biggest one, we are interested in decompose the convex hull using the smallest number of empty interior disjoint convex polygons, with vertices in the given point set. Specifically, we give an upper bound on the number of these polygons. Such set of polygons is called *Convex Decomposition*.

For instance, a triangulation is a convex decomposition in which every element, obviously, is a triangle. The number of triangles obtained is well known. If we could decompose in quadrilaterals, the number of elements, would be the number of triangles divided by two; if we could decompose in pentagons, the number of elements would be the number of triangles divided by three, etcetera. But its easy to find point sets that do not admit a convex decomposition containing only quadrilaterals (and hence only pentagons, or only hexagons, and so on). So we are allowing different polygons.

It is known that, in any triangulation always we can find pairs of triangles that can be joined to obtain convex quadrilaterals. We use this idea to obtain bigger convex polygons, such pentagons or hexagons, and get a better upper bound.

By the nature of the problems we study in this work, we remark that, all the point sets we consider have no three or more elements on a line.

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Chapter 1

Introduction

Paul Erdős said that Computational Geometry started in 1694, with the problem of how many unit spheres can be touched by a single unit one. Isaac Newton believed that the number was 12 while David Gregory claimed it was 13. This problem was solved until 1953 by Shütte and van der Waerden. They proved that Newton was right.

Computational Geometry and Discrete Geometry have grown considerably. The success of this fields can be explained by itself from the beauty of the problems and its solutions, and the lots of its applications. Among its applications we can mention: illuminating, guarding art galleries, image processing, computational complexity, etc. Topics on which there is a clear symbiosis between Mathematics and Computer Sciences.

Its worthy to mention about the recent topic that appeared, making even more clear this symbiosis: the Computational Complexity, which makes the analysis of an algorithm and it, roughly speaking, says how many steps inevitably a computer has to make, when an algorithm is implemented by a good or a bad programmer. And then appeared the NP-complete problems, which are so complicated that the computer will not be useful despite its power of processing. In this problems it is very important the mathematical tools used to calculate the number of steps, that there will be in solving a case of a problem.

About the beauty, for mention something, there is the Paul Erdős phrase: "God keeps the perfect proofs for the mathematical theorems in The Book". M. Aigner and G. M. Ziegler continued the work of Paul Erdös of selecting problems, and solutions, that on their opinion (and the lots of us) must be in The Book. We can find several theorems with its proofs, in Discrete and Computational Geometry, in the compilation "*Proofs from THE BOOK*" [2].

We state a precedent in Discrete Geometry, proved by Esther Klein in 1931, interesting for our work. She observed that in any set of five points on the plane, there is always a convex quadrilateral with its vertices in such set. We state it formally with the following theorem:

Theorem 1 *(E. Klein, 1931) In any set of 5 points, there is always a set of four of them being the vertices of a convex quadrilateral.*

Proof: Trivially if the points are the vertices of a convex pentagon or a quadrilateral (with one point in its interior), we are done. See Figure 1.1.

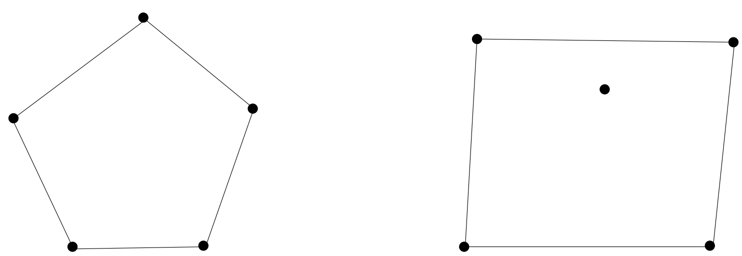

Figure 1.1: Trivial cases.

In the case that the points are the set of vertices of a triangle T with two points in its interior, call such interior points p and q, and make ℓ the line pq. We will have that the two vertices of T on the same side of ℓ, and p and q are the vertices of a convex quadrilateral. See Figure 1.2.

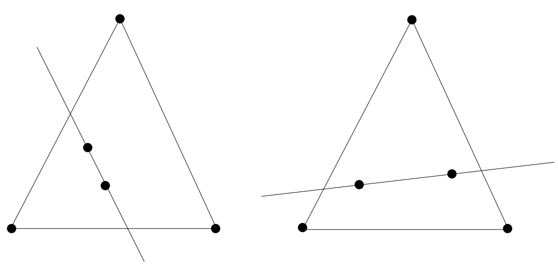

Figure 1.2: Quadrilateral with vertices p and q, and the vertices of the triangle on the same side of the line pq.

∎

In this work we are interested in two problems of Discrete Geometry: *The chromatic number of the disjointness graph*, a geometric variant of Knesser Graph, and the problem of *convex decompositions*. We have two definitions we use to describe this work.

Definition 2 *Let P be a point set in the plane. We say that P is in* general position *if there are no three or more elements of P on a straight line.*

Definition 3 *Let P be a point set in the plane. We say that P is in* convex position *if P is in general position and is the set of vertices of a convex polygon.*

Problem 1. The first problem we study here will be described as follows: We take P, a set of n points in the plane, and for every pair of points we draw the line segment joining them. We will color all the segments according to the following rule: if the closures of any two given segments have no common points, then they must have different color. We are interested in the smallest number of colors needed to do it.

The problem is only solved when P is in convex position. Here we study the number of colors when the point set is the double chain. We prove that the number of colors required is substantially bigger than those needed when the vertex set is in convex position.

If P is not in convex position, still we can find a convex polygon C having elements of P as its vertices, and covering the remaining points. We give a definition.

Definition 4 *The* convex hull *of P is the smallest convex set containing P. We denote it as* $\mathrm{conv}(P)$*. Also we will denote as $h = h(P)$ the number of its vertices.*

Problem 2. The second problem we study is about Convex Decompositions. In this problem, quite the opposite to the previous one, we will add the fewest necessary edges as possible and shatter (decompose) the convex hull of P in interior–disjoint convex polygons, such as triangles, quadrilaterals, pentagons, etc, with its vertices on P. Such set of polygons is called convex decomposition. We will show how to obtain a convex decomposition of P with at most $\frac{10}{7}|P|$ elements.

Esther Klein's Theorem is interesting in Problem 1, since we can try to generalize it to find the biggest subset S of P in convex position (having or not other points of P in its interior). We will color optimally the segments with both ends on S, and color the remaining segments with another strategy. See Figure 1.3.

Esther Klein's Theorem is also interesting in Problem 2, since it takes us to find convex polygons, but this time, empty. See Figure 1.3. We describe these ideas and give the formal definitions in next chapters.

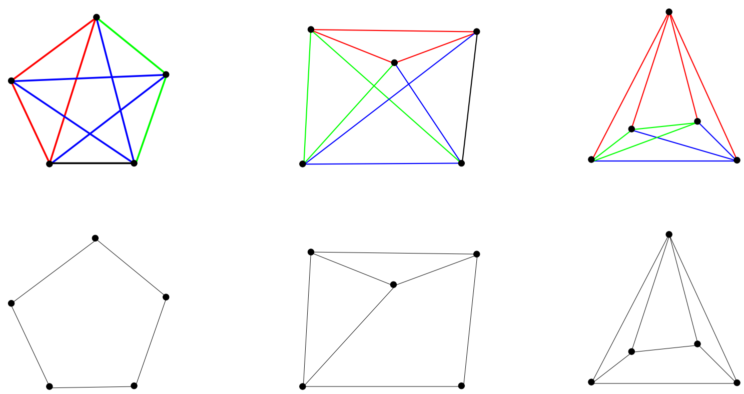

Figure 1.3: Different drawings of K_5 colored, and convex decompositions of the different 5–point sets.

By the nature of the problems we study in this work, as we mentioned, we will always assume that the point sets under consideration are in general position. Also, instead of talking about that a point set which contains the set of vertices of a convex r-gon, we simply say that it contains a convex r–gon. Obviously, such polygon will be empty if it has no other points in its interior.

As in both problems we study here, the number of colors and the number of convex polygons, are in terms of the cardinality of the point set under consideration, P will denote a finite set of points in the plane in general position. Also n will denote its cardinality, always with $n \geq 5$.

1.1 Outline of this thesis

In next chapter we give notions of the tool we are using: *Graph Theory*. We give the basic definitions, and the way we are representing the graphs so we can use them.

In Chapter 5 we analyze the chromatic number of the disjointness graph, induced by the resulting drawing with vertices in the double chain.

In Chapter 6 we continue studying convex polygons. Not only the biggest one in the point set. We will allow others as triangles, pentagons, etc.

Finally, in Chapter 7 we state the future work.

Chapter 2

Graphs

We are using graphs in this work. Here is the formal definition [5].

Definition 5 *A* graph *is a pair $G = (V, E)$ such that $V \neq \emptyset$ and $E \subseteq V^2$, thus the elements of E are 2–elements subsets of V. We shall always assume that $E \subseteq V^2 \setminus \{(v, v) : v \in V\}$. The elements of V are the* vertices *of the graph G, and the elements of E are its* edges*.*

Note. In this thesis we will work only with finite sets of vertices.

If $e = (u, v) \in E$, we say that e *joins* the vertices u and v, and we denote it as $uv = vu$. We say u and v are *adjacent*, and that such edge is *incident* with both u and v, or that u and v are its *endvertices* or its *ends*.

So, we shall not always distinguish strictly between a graph $G = (V, E)$ and its vertex or edge set. We will speak of a vertex $v \in G$ rather than $v \in V$, and edge $e \in G$ rather that $e \in E$, and so on. And given a point (vertex) v, not in G, we write $G \cup v$ instead of $V \cup \{v\}$, or if v is a vertex in G, we write $G \setminus v$ instead of $V \setminus \{v\}$, the same as with an edge e, not in G, we write $G \cup e$ rather than $E \cup \{e\}$ and, if e is in G, we write $G \setminus e$ rather than $E \setminus \{e\}$.

A *path* is a non-empty graph $W = (V, E)$ of the form $V = \{v_0, v_1, v_2, ..., v_k\}$ and $E = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\}$, where all the v_i are distinct. We say that v_0 and v_k are *linked* by W and are called its *endvertices* or *ends*; $v_1, v_2, ..., v_{k-1}$ are the *inner vertices* of W. The number of edges of a path is its *length*. We often refer to a path by the natural sequence of its vertices writing $W = v_0v_1v_2...v_k$. And, if for any given u, v in a graph G there is always a path linking u and v, we will say that G is *connected*.

If $W = v_0v_1v_2...v_k$ is a path, and v_0 and v_k are adjacent, then the graph $C = W \cup v_kv_0$ will be called a *cycle*. As we do in paths, we denote a cycle by its cyclic sequence of vertices, so C might be written as $v_0v_1v_2...v_kv_0$.

These abstract objects are called graphs since they can be represented graphically on the plane keeping all its properties [4]. The pictures representing them are called drawings.

A *drawing* of a graph G in the euclidean plane consists of a point set, one for each vertex of G, and a collection of simple open arcs, one for each edge, such that if e is an edge of G with ends u and v, then the topological closure of the arc α representing e consists of α and the points representing u and v.

We say that a drawing is *good* if:

i) Any two arcs (edges) have only finitely many points in common.

ii) If two arcs have a common interior point, then they properly cross at this point, that is, the first arc passes from one side of the second arc to the other side.

iii) No three arcs have a common interior point.

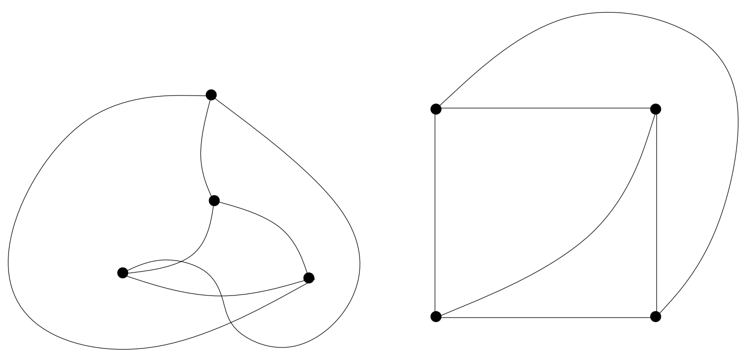

Figure 2.1: Good drawings of the same graph.

Note. We are interested only in good drawings, so from now on, when we say drawing we mean good drawing.

Observe that in the definition of good drawing, we allow intersections between the edges.

Definition 6 *Let $\mathcal{D}$ be a drawing of the graph G. A* crossing *in $\mathcal{D}$ is a point in $\mathbb{R}^2$ in which two edges intersect. The number of crossings in $\mathcal{D}$ will be denoted as*

$$cr(\mathcal{D}).$$

The concept of crossing is fundamental for Problem 1, since we will be comparing, in all pair of edges, if they cross each other, if they have a vertex in common or if they are disjoint. In Problem 2 we want drawings strictly without crossings.

Definition 7 *A graph G is* planar *if there exists a drawing of it in which there are no intersections between its edges. Equivalently, G will be called* planar *if, and only if there exists a drawing $\mathcal{D}$ of G with $cr(\mathcal{D}) = 0$.*

Observe that in Figure 2.1 there are 3 crossings in the left drawing, while in the right one there are no crossings, so the graph drawn is planar.

Let G be a planar graph and $\mathcal{D}$ be a drawing of G without crossings. A connected region of $\mathbb{R}^2 \setminus \mathcal{D}$ will be called a *face*. As every graph considered in this work is a finite set, all its drawings are compact, so if $\mathcal{D}$ is a drawing, there is an open disc F containing $\mathcal{D}$. The infinite region containing $\mathbb{R}^2 \setminus F$ will be called the *infinite face* or the *external face*. The finite faces will be called *internal faces*.

By the nature of the problems we study, we work only with *geometric graphs*: graphs drawn on the plane with its edges represented as straight line segments. These representations are also called *rectilinear drawings*. In the next figure we show the rectilinear drawings of the graphs drawn on Figure 2.1.

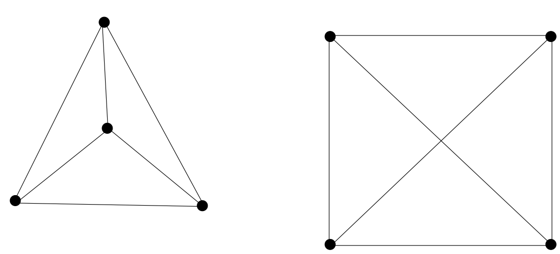

Figure 2.2: Rectilinear drawings of the same graph.

Definition 8 *The* complete graph *with vertex set P is the graph $K_n = (P, E)$ in which $E = \{uv : u, v \in P$ and $u \neq v\}$.*

Two edges will cross each other, in any drawing of K_n, if and only if they are the diagonals of a convex quadrilateral. See Figure 2.2. By Theorem 1, any drawing of K_n will have crossings for $n \geq 5$. And as we mentioned, we are interested in whether or not two edges have a point in common so they can be colored. In Figure 2.2 we see the importance of the location of the points.

We give a trivial bound for Problem 1:

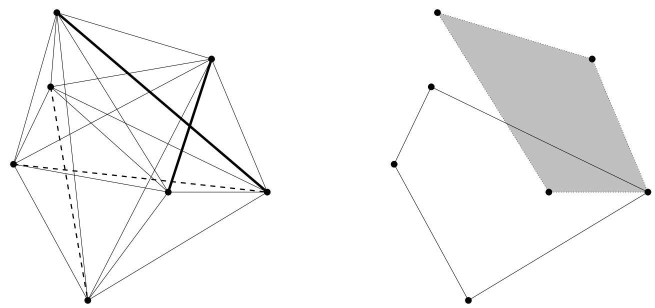

Figure 2.3: Drawing of K_7. The crossing between shaded edges is induced by the gray quadrilateral. The one with dotted edges is induced by the white quadrilateral, and so on: there is a crossing if and only if there is a convex quadrilateral.

Proposition 9 *The number of colors needed to color the edges of K_n, as described in Problem 1, is at least*

$$\left\lceil \frac{n}{2} \right\rceil.$$

Proof: Since P is in general position, then it is easy to find a plane cycle $C = v_1 v_2 \dots v_n v_1$ with vertex set P. And as $v_i v_{i+1}$ and $v_j v_{j+1}$ are disjoint, for every $2 \le i+1 < j \le n-1$, then every three consecutive edges of C cannot have same color, so, the set of edges of C needs at least $\lceil \frac{n}{2} \rceil$ colors. ∎

Next we give a trivial bound for Problem 2.

Quite the opposite, in convex decompositions we want as few edges as possible. We want to obtain a planar connected graph, with the fewest possible convex faces shattering the convex hull of P. We mentioned that a triangulation is a convex decomposition. In fact, it is the convex de composition with the biggest number of elements, namely $2n - h - 2$. We can improve this bound, trivially, as we show next. We use the definitions in [10, 16]:

Definition 10 *Let T be a triangulation of P. If e is the common edge of two triangles $t_1, t_2 \in T$ such that $Q = t_1 \cup t_2$ is a convex quadrilateral, then we say that e is a* flippable edge. *By* flipping *e we mean the operation of removing e and replacing it by the other diagonal of Q.*

Observe that, if we have a convex k–gon Q triangulated in a triangulation T, the diagonals of Q are flippable edges. So, if we remove them from T, we decrease the number of polygons. But, in general, the converse situation of removing flippable edges, being sides of same triangle, in any given triangulation,

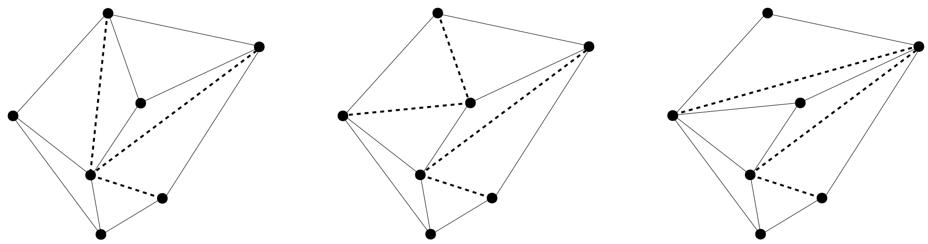

Figure 2.4: Triangulations with its flippable edges dashed. The triangulations on the left and the central one are obtained by means of a flip. Similarly, the central and the one on the right.

cannot result a convex polygon. See the triangulation on the center in Figure 2.4. This situation lead us to the following definition.

Definition 11 *Let T be a triangulation of P, with e and e' two flippable edges. If e and e' are not sides of the same triangle, we will say that they are* simultaneously flippable edges*.*

J. Galtier, F. Hurtado, M. Noy, S. Perennes, J.Urrutia [10] prove that in any triangulation there are always a set with at most $\frac{n-4}{6}$ edges that can be simultaneously flipped. If we remove such edges, we will obtain a convex decomposition with $2n-h-2-\frac{n-4}{6}=\frac{11n-8}{6}-h$ elements: $\frac{n-4}{6}$ quadrilaterals and $\frac{5n-2}{3}-h$ triangles. We have proved the following bound:

Theorem 12 *There exists a convex decomposition of P with at most*

$$\frac{11n-8}{6}-h$$

elements.

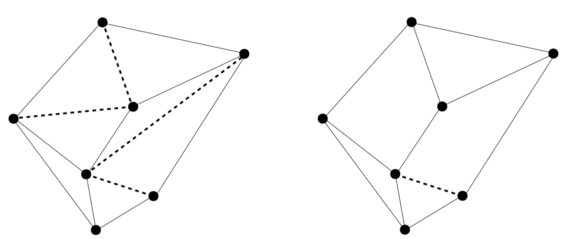

Figure 2.5: Convex decomposition after removing one pair simultaneous flippable edges.

To find our bound, we will start from a specific triangulation and use the idea of removing edges.

Finally, we give some definitions we are using in the next chapter.

Definition 13 *Let $G = (V, E)$ be a graph. A* vertex coloring *of G is an assignment of k colors to the vertices of G. The coloration is* proper *if every two adjacent vertices have different colors.*

That is, given a set K of colors, a proper vertex coloring of G is a function $c : V \to K$ such that if $uv \in E$ then $c(u) \neq c(v)$. See Figure 2.6. We denote by $|c(G)|$ the number of colors in K used to color G.

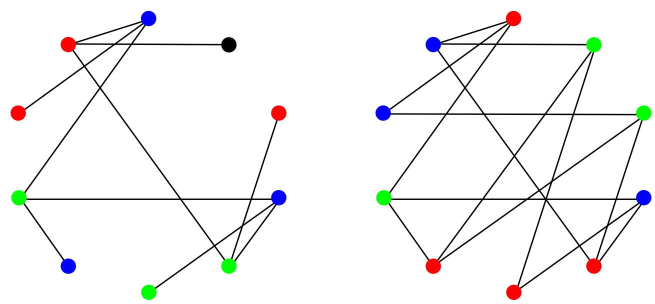

Figure 2.6: Proper vertex coloring of two graphs: adjacent vertices must be colored with different colors.

The *chromatic number* of a graph G is the smallest number of colors needed to get a proper vertex coloring of G. We denote it as $\chi(G)$. A coloring $c(G)$ will be optimal if $|c(G)| = \chi(G)$. In Figure 2.6 we colored two graphs, the one on the left can be colored using only three colors, by coloring in blue or green the vertex colored in black. Observe that the chromatic number of both graphs is 3.

Chapter 3

Coloring the edges in K_n with its vertices in the double chain

In this chapter we present the results obtained in [8].

We obtain the rectilinear drawing $\mathcal{D}(P)$ of K_n induced by P, and color all its edges with the only restriction that: for any given two edges, if their respective closed segments are disjoint, then they must have different colors.

We explain this situation with graph coloring.

3.1 The disjointness graph

Definition 14 *Let $P = \{v_1, v_2, ..., v_n\}$ and $\mathcal{D}(P)$ be the rectilinear drawing of K_n with vertices in P. We define the* disjointness graph *$D(P)$ as the graph which vertex set is $V = \{ij : 1 \leq i < j \leq n\}$, with two vertices, ij and $i'j'$, being adjacent if and only if the corresponding closed segments, $\overline{v_i v_j}$ and $\overline{v_{i'} v_{j'}}$, are disjoint in $\mathcal{D}$. See Figure 3.1.*

We will look for an optimal proper vertex coloring of the disjointness graph. That is, given P and the rectilinear drawing $\mathcal{D}(P)$ of K_n, we will obtain $D(P)$, see Figure 3.1. Now we obtain an optimal vertex coloring $c(D(P))$ in order to color the edges in $\mathcal{D}(P)$ according on $c(D(P))$. See Figure 3.2.

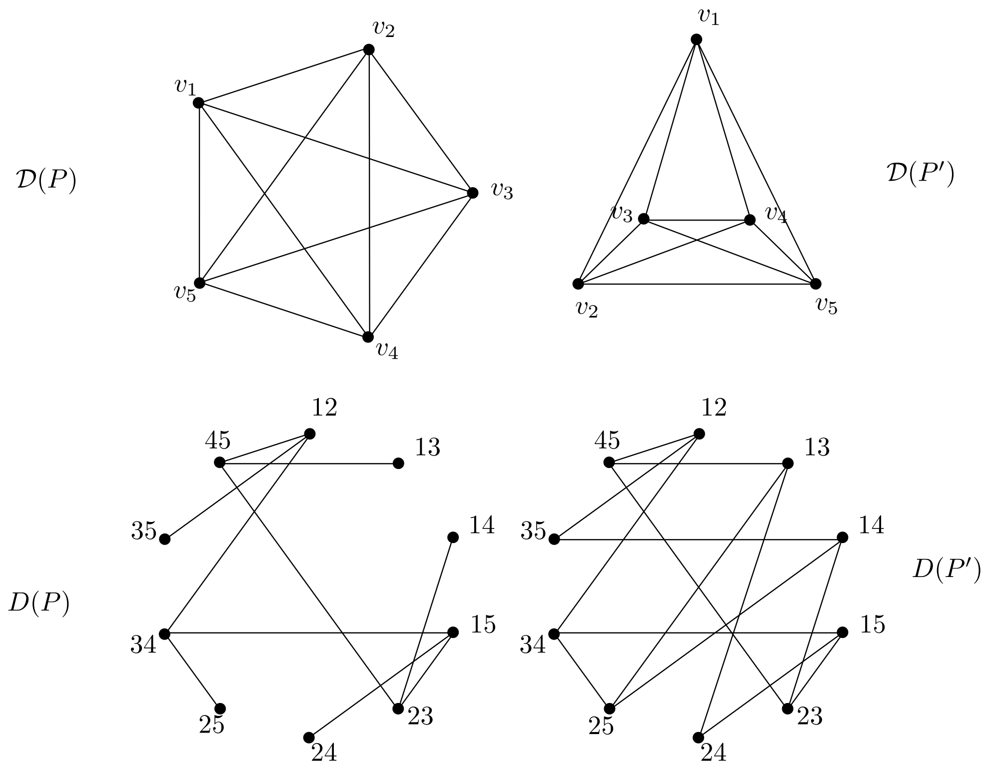

Figure 3.1: Given the point sets with 5 elements P and P', we obtain the respective rectilinear drawing of K_5, $\mathcal{D}(P)$ and $\mathcal{D}(P')$, then we obtain the disjointness graph $D(P)$ and $D(P')$.

3.2 Coloring the edges in the drawing.

Let c be a proper vertex coloring of $D(P)$. We will abuse notation by denoting as $c(P)$ the coloring of the drawing $\mathcal{D}(P)$, instead of $c(D(P))$. Also for $Q \subset P$ we will use $c(Q)$ to denote the edge coloring, of the complete graph with vertices in Q, induced by c. For an edge e, $c(e)$ will denote the color that c assigns to e. We will denote by $|c(P)|$ the number of colors in c. In Figure 1.3 we show proper colorations of the three different drawings of K_5. In the following, by coloration we will mean proper coloration.

We will say that the coloration c, of P, is *optimal*, if $|c(P)| = \chi(D(P))$, that is, if $|c(P)| = \min\{|c(P)| : c \text{ is a coloration of } P\}$. In general, the bounds for the number of colors in any optimal coloration among all n–point sets are:

$$L(n) := \min\{|c(P)| : P \text{ is an } n \text{ point set and } c \text{ is optimal}\}$$

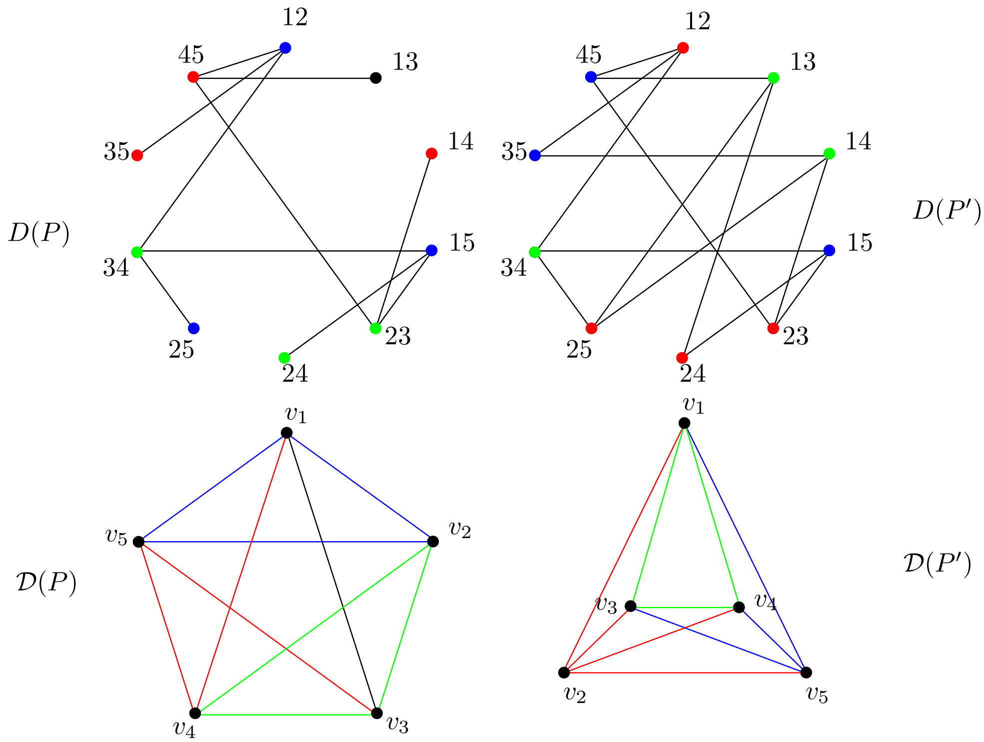

Figure 3.2: Every coloring of $D(P)$ and $D(P')$ induces a proper edge coloring on the drawing $\mathcal{D}(P)$ and $\mathcal{D}(P')$ respectively.

and

$$U(n) := \max\{|c(P)| : P \text{ is an } n \text{ point set and } c \text{ is optimal}\}.$$

The following bounds for $U(n)$ were established in [3]:

$$5\left\lfloor \frac{n}{7} \right\rfloor \leq U(n) \leq \min\left\{n-2, n+\frac{1}{2}-\frac{\lfloor \log\log n \rfloor}{2}\right\}.$$

In Figure 3.3 we show optimal colorations of K_6 for each point set.

For the case in which P is in convex position, the exact value of any optimal coloration c of P is known. The upper bound is given by J. Jonsson [17]. Later R. Fabila and D. Wood [9] prove that this is in fact the lower bound. Write

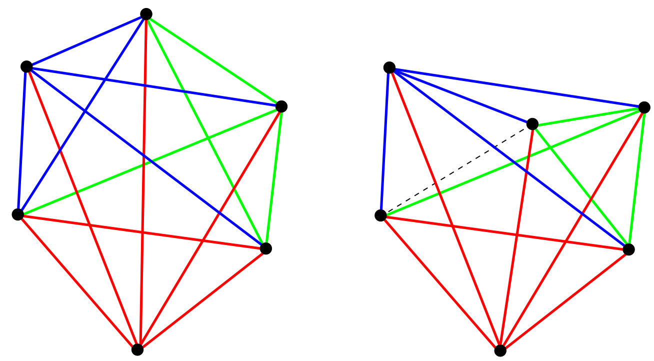

Figure 3.3: Drawings of K_6 colored optimally. Observe that the dotted edge cannot be colored red, blue or green.

$$f(n) := n - \left\lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rfloor . \tag{3.1}$$

As far as we know, the family $\mathfrak{C}$ of the points sets in convex position is the only for which the exact value of any optimal coloration is known for every $P \in \mathfrak{C}$. Our goal in this thesis is to determine the exact value of $|c(Q)|$ for Q, an element of the family $\mathfrak{V}$, of the double chain points sets. We remark that if $Q \in \mathfrak{V}$ and c is an optimal coloration of Q, then $|c(Q)|$ can be signifficatively bigger than $f(n)$.

An integer function will be very used in this paper is the following:

$$g(n) := \max \left\{ i : i \in \mathbb{Z}^+, \binom{i}{2} \leq n \right\} . \tag{3.2}$$

Unless otherwise is stated, for the rest of this thesis, $f(n)$ is as in (3.1) and $g(n)$ is as in (3.2).

As we mentioned, this problem was introduced first by Araujo, Dumitrescu, Hurtado, Noy and Urrutia in [3], as geometric analog of the *Kneser graph*. We recall that the Kneser graph $KG(n;k)$ has as vertices all the k-subsets of a set of n elements; two vertices are adjacent if their respective k-subsets are disjoint. Obviously, in our version, we work with $k = 2$, so $U(n)$ and $L(n)$ will be stated as

$$L(n) = \min\{\chi(D(P)) : P \text{ is an } n \text{ point set}\}$$

and

$$U(n) := \max\{\chi(D(P)) : P \text{ is an } n \text{ point set}\}.$$

A chromatic class S of c is a *star* if its corresponding edges in $\mathcal{D}$ share a unique vertex, which is called the *apex* of S. If S consists of exactly one edge, then any of its endpoints can be the apex of S. If v is a vertex of P and S is a star of c having v as an apex, then we will say that v is an *apex* of c. Any chromatic class of c that is not a star, is a *thrackle*. See Figure 3.4.

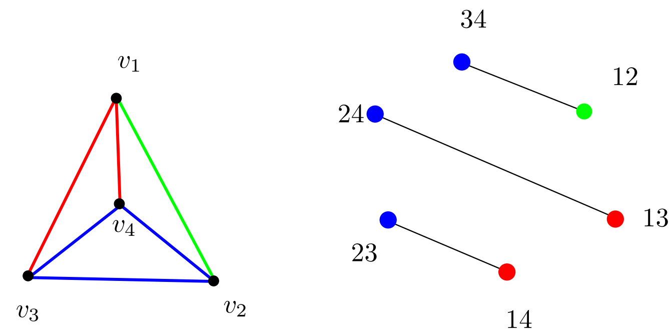

Figure 3.4: Coloring of a drawing of K_4 induced by a coloring of its disjointness graph. We find a blue thrackle and two stars.

3.3 The value of $\chi(D(C_n))$

We will denote as C_n to the set of vertices of a convex n-gon and we will assume that its elements are labelled as v_1, v_2, ..., v_n, consecutively around $\text{conv}(C_n)$. We give the construction, due to Jonsson [17], to get an optimal coloring of the edges in $\mathcal{D}(C_n)$.

Let i, j be integers with $1 \leq i < j \leq n$. A *polyomino* is a finite subset of $\mathbb{Z}^2$. For later convenience, we adopt the matrix convention for indexing rows and columns in $\mathbb{Z}^2$; row i is just below row $i-1$, column j is just to the right of column $j-1$, and ij refers to the lattice point in row i and column j. We identify the edge v_iv_j with the lattice point ij, which we represent as a unit square. In this manner, we may represent all the edges in K_n in the polyomino as illustrated on the left in Figure 3.5.

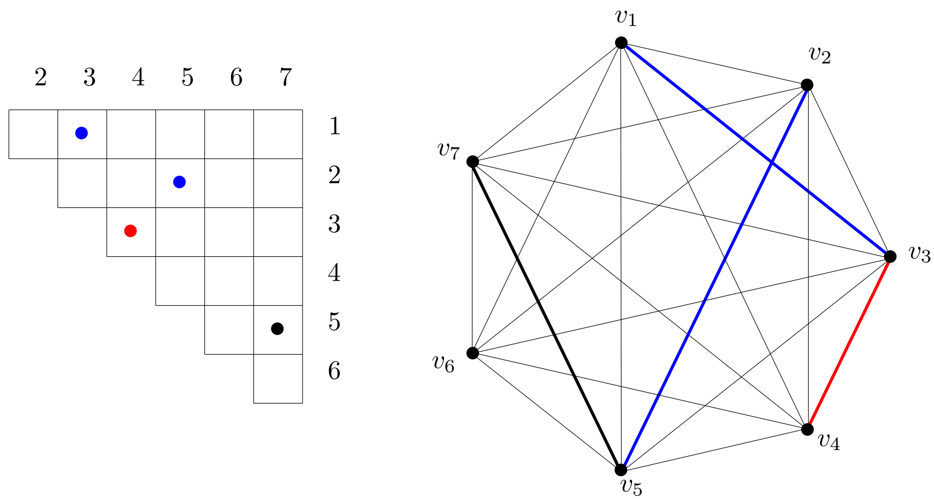

Figure 3.5: Diagram representing the edges of K_7. The ij cell on the diagram represents the edge v_iv_j on the drawing, always $1 \le i < j \le n$.

We will have that any two distinct edges v_iv_j and $v_{i'}v_{j'}$ are disjoint if

i) $j' < i$,

ii) $j < i'$,

iii) $i' < i < j < j'$ or

iv) $i < i' < j' < j$.

See Figure 3.6 left. The edges colored gray, green, blue and red are as in case i), ii), iii), and iv) respectively. Equivalently, v_iv_j and $v_{i'}v_{j'}$ have a common point if and only if

i) $i \le i' \le j \le j'$ or

ii) $i' \le i \le j' \le j$.

See Figure 3.6 right. The dashed and dotted edges are as in case i) and ii) respectively.

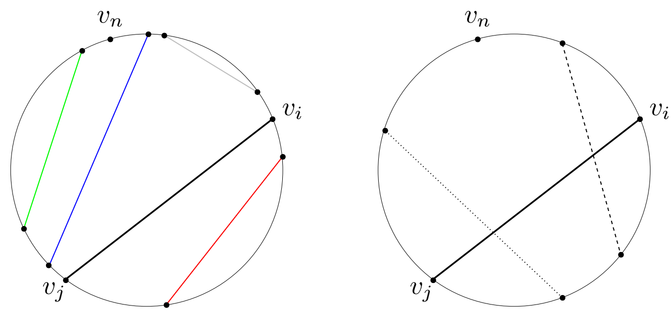

Figure 3.6: Left. There are four cases (colored gray, green, blue and red) where the edges are disjoint from the (black) edge v_iv_j. Right. There are two cases of edges (dashed and dotted) having a common point with the (black) edge v_iv_j.

Two edges v_iv_j and $v_{i'}v_{j'}$ are in different chromatic class if $i'j'$ lies in the colored regions in the diagram shown on Figure 3.7 left. Equivalently, two edges v_iv_j and $v_{i'}v_{j'}$ can be in the same chromatic class, if $i'j'$ lies in the colored region in the diagram shown in Figure 3.7 right.

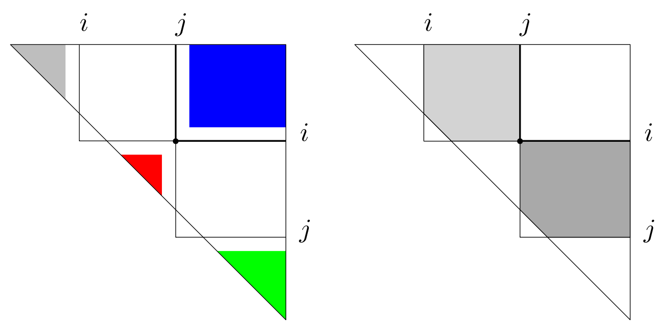

Figure 3.7: Left. The gray, green, blue and red regions keep the cells in different chromatic classes from the ij cell. Even more, the respective cells of the gray, green, blue and red edges, in Figure 3.6, lie in the respective colored region. Right. There are two regions where the cells can be on the same chromatic class of ij. Even more, the respective cells of the dotted and the dashed edges, in Figure 3.6, lie in the lightgray and the darkgray region respectively.

Clearly, the bigger chromatic classes, the fewer colors we use.

Erdős proved in [7] that every maximal thrackle with n vertices has at most n edges. In the polyomino, every thrackle with n edges will be represented as a path of n consecutive cells going only downwards or rightwards starting and finishing in the same value i, $2 \leq i \leq n-1$. See Figure 3.8.

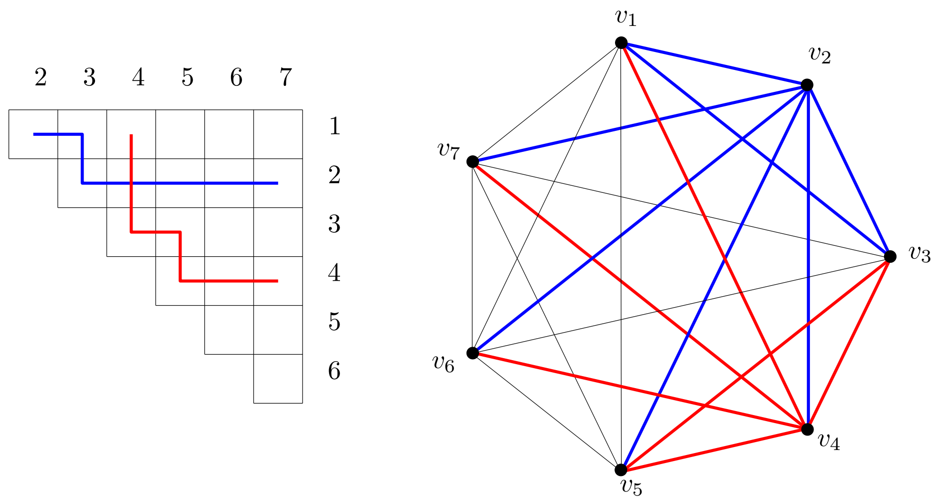

Figure 3.8: The red and blue paths on the diagram represent in $\mathcal{D}(C_7)$ a red and blue thrackle respectively. We can see that we need two more colors to color all the remaining edges.

Observe that any two paths in the diagram share at least one cell. Hence we have that k paths cover at most $kn - \binom{k}{2}$ cells. Using the smallest number of paths as possible, to cover all cells, we must have that

$$kn - \binom{k}{2} \geq \binom{n}{2}.$$

Solving the quadratic equation, we get that $k \geq f(n)$. We state several elementary, but useful, results for the rest of the work.

Proposition 15 *Let i, m, and r be positive integers with $1 \leq r < i$.*

(*I*) $f(m) = m - g(m) + 1$.

(*II*) $m = \binom{i}{2}$ *if and only if* $f(m) = f(m-1)$.

(*III*) *If* $m = \binom{i}{2} + r$, *then* $f(m-1) = f(m) - 1$.

Proof: We show separately these assertions.

(*I*) The required equation is Theorem 1.1 in [17].

(*II*) Since $m = \binom{i}{2}$, then $g(m) = i$ and $g(m-1) = i-1$. Using these equalities in (*I*) we get, respectively, $f(m) = m - i + 1$ and $f(m-1) = (m-1) - (i-1) + 1 = m - i + 1$. Therefore, $f(m) = f(m-1)$.

Conversely, let $j = g(m-1)$ and $i = g(m)$. By definition of g, we have that $\binom{j}{2} \leq m-1 < \binom{j+1}{2}$ and $\binom{i}{2} \leq m$. From (*I*) and our hypothesis $f(m) = f(m-1)$, it follows that $j+1 = i$. Substituting this in $m-1 < \binom{j+1}{2}$ we obtain that $m-1 < \binom{i}{2} \leq m$. So it must be that $m = \binom{i}{2}$.

(*III*) From $m = \binom{i}{2} + r$ and $1 \leq r < i$ it follows that $g(m) = i = g(m-1)$. Again, these equalities and (*I*) imply, respectively, that $f(m) = m - g(m) + 1$ and $f(m-1) = (m-1) - g(m-1) + 1 = m - g(m)$. Therefore, $f(m-1) = f(m) - 1$.

∎

Our next result is a direct consequence of (*II*) and (*III*) in Proposition 15.

Lemma 16 *Let i be a positive integer. Then*

$$f(n+1) = \begin{cases} f(n) & \text{if } n = \binom{i}{2} - 1 \\ f(n)+1 & \text{otherwise} \end{cases}$$

and hence $f(n+k) \leq f(n) + k$, for any nonnegative integer k.

Proposition 17 *Let c be an optimal coloring of $D(P)$. If $S_1, \ldots, S_r$ are distinct stars of c, then they have distinct apexes.*

Proof: For $i \in \{1, ..., r\}$, let v_i be the apex of S_i. Seeking a contradiction, suppose that there are distinct $i, j \in \{1, ..., r\}$ such that $v_i = v_j$. Thus $S_i \cup S_j$ can be considered as a star with apex v_i, and it can be colored with one color, producing a new coloring of $D(P)$ with less colors than $|c|$, which contradicts the optimality of c. ∎

Proposition 18 *Let c be an optimal coloring of $D(P)$, and let $S_1, \ldots, S_r$ be r distinct stars of c with respective apexes $v_1, \ldots, v_r$. Then for $i \in \{1, ..., r\}$, we can extend every S_i until it reaches $n - i$ edges in such a way that the resulting coloring c' of $D(P)$ is also optimal.*

Proof: For $i \in \{1, ..., r\}$, let e_i be an edge of S_i. We obtain c' by modifying c as follows: color with $c(e_r)$ all the edges of $\mathcal{D}(P)$ which are incident with v_r, next color with $c(e_{r-1})$ all the edges of $\mathcal{D}(P)$ which are incident with v_{r-1}, and so on. From Proposition 17 this procedure is well defined. Clearly, the resulting coloration c' of $D(P)$ is proper and has at most $|c|$ colors. In particular, note that in c' the number of edges of $\mathcal{D}(P)$ having color $c(e_i)$ is exactly $n - i$. ∎

Proposition 19 *Let c be an optimal coloring of $D(P)$. If $S_1, \ldots, S_r$ are distinct stars of c, and for $i = 1, \ldots, r$, v_i is an apex of S_i, then*

$$\chi(D(P \setminus \{v_1, \ldots, v_r\})) = \chi(D(P)) - r.$$

Proof: By Proposition 17 we may assume that $v_i \neq v_j$ for any $i, j \in \{1, ..., r\}$ with $i \neq j$, and by Proposition 18 that S_1 has $n-1$ edges. Then, if we delete v_1 and all its incident edges, the remaining graph $D(P \setminus \{v_1\})$ no longer has edges of S_1. So $|c(D(P \setminus \{v_1\}))| \leq |c(D(P))| - 1 = \chi(D(P)) - 1$.

Now, if $|c(D(P \setminus \{v_1\}))| < \chi(D(P)) - 1$, then there exists a coloration c' of $D(P \setminus \{v_1\}))$ with at most $\chi(D(P)) - 2$ colors. Thus c' together with S_1 give us a coloration of $D(P)$ with at most $\chi(D(P)) - 1$ colors, which again contradicts the optimality of c. Hence $|c(D(P \setminus \{v_1\}))| = \chi(D(P)) - 1$.

By repeating the previous reasoning for $v_2, \ldots, v_r$ we can establish

$$\chi(D(P \setminus \{v_1, v_2, \ldots, v_r\})) = \chi(D(P)) - r,$$

the desired equality. ∎

Proposition 20 *Any optimal coloration of $D(C_n)$ has no stars if and only if $n = \binom{g(n)}{2}$.*

Proof: For brevity, let $i := g(n)$. Assume that every optimal coloration of $D(C_n)$ has no stars. Let c be an optimal coloration and suppose that $|c(D(C_n \setminus \{u\}))| < |c(D(C_n))|$, for some $u \in C_n$. From $c(D(C_n \setminus \{u\}))$ we can obtain a coloration c' of $D(C_n)$ by adding a star with apex u such that $|c'| \leq |c|$. This contradicts the hypothesis that there are no optimal colorations with stars. So we must have that $|c(D(C_n))| = |c(D(C_n \setminus \{u\}))|$ for any $u \in C_n$. That is, $f(n) = f(n-1)$. Again, this and Lemma 16 imply that $n = \binom{i}{2}$.

Now we suppose that $n = \binom{i}{2}$. Let c be an optimal coloration of $D(C_n)$, and suppose that it has a star S with apex s. Then $|c(D(C_n \setminus \{s\}))| = f(n) - 1$, by

Propositions 19. Since $D(C_{n-1}) = D(C_n \setminus \{s\})$, then $|c(D(C_n \setminus \{s\}))| = f(n-1)$, and hence $f(n) - 1 = f(n-1)$. On the other hand, from Lemma 16 we know that $f(n) = f(n-1)$, which contradicts previous equality. Therefore, c cannot have stars.

∎

Proposition 21 *If* $r = n - \binom{g(n)}{2}$, *then the number of stars in any optimal coloring of* $D(C_n)$ *is at most* r.

Proof: The case $r = 0$ is given in Proposition 20. So we may assume that $r > 0$. We derive a contradiction from the assumption that there exists an optimal coloration c of $D(C_n)$ containing $r+1$ distinct stars, say $S_1, \ldots, S_{r+1}$. By Proposition 18 we may assume that S_i has exactly $n - i$ edges, for $i \in \{1, ..., r, r+1\}$. In particular, observe that S_{r+1} has $n - r - 1 = \binom{g(n)}{2} - 1 \geq 2$ edges.

Let $W = \{v_1, \ldots, v_r\}$ be the set of apexes of $S_1, \ldots, S_r$. By Proposition 19 we know that

$$\chi(D(C_n \setminus W)) = \chi(D(C_n)) - r.$$

The last equality and the fact $|c(D(C_n \setminus W))| \leq \chi(D(C_n)) - r$ imply that $c(D(C_n \setminus W))$ is an optimal coloration for $D(C_n \setminus W)$. On the other hand, from the facts $C_{n-r} = C_n \setminus W$ and $n - r = \binom{g(n)}{2} = \binom{g(n-r)}{2}$, and the Proposition 20, we can conclude that $c(D(C_n \setminus W))$ cannot have any star. This contradicts that S_{r+1} is a star of $c(D(C_n \setminus W))$. ∎

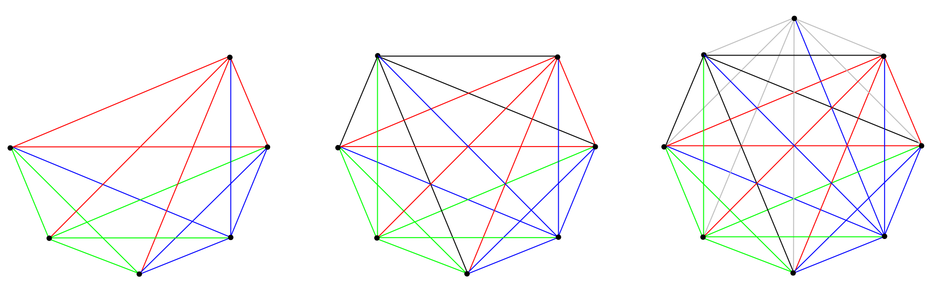

Figure 3.9: Optimal colorings for K_6, K_7, and K_8 with r stars. Here $i = 4$, so $n = 6 + r$, for $r = 0, 1, 2$.

For the nex result we will use the following theorem.

Theorem 22 (Theorem 2 in [9]) *For every set P of n points in convex and general position, the union of k maximal thrackles on P has at most*

$$kn - \binom{k}{2}$$

edges.

Proposition 23 *Any optimal coloration of $D(C_n)$ has at most one chromatic class consisting of a single vertex.*

Proof: Seeking a contradiction, suppose that C_n is a counterexample of minimum order. Then there exists an optimal coloration c of $D(C_n)$ which contains two chromatic classes S_1 and S_2 such that $S_1 = \{v_1\}$ and $S_2 = \{v_2\}$ where v_1 and v_2 are distinct vertices of $D(C_n)$. In other words, for $i \in \{1, 2\}$ we have that $c(v_i) \neq c(u)$ for any vertex u in $D(C_n) \setminus \{v_i\}$.

The minimality of C_n and Proposition 19 imply that S_1 and S_2 are the only stars of c. As $|c(C_n)| = f(n)$, and that there are only two stars, the number of thrackles in c will be $k = f(n) - 2$. Let $T_1, \ldots, T_k$ be the set of all such thrackles. From Theorem 22, we know that the number of vertices of $D(C_n)$ in $T_1 \cup \cdots \cup T_k$ is at most $kn - \binom{k}{2}$. Since $T_1, \ldots, T_k, S_1$ and S_2 are all the chromatic classes of c, we have that $kn - \binom{k}{2} + 2 \geq \binom{n}{2}$.

From previous inequality, we get that $(n-k)^2 \leq n+k+4$. From Proposition 15 (*I*) we know that $k = n-g(n)-1$. Substituting this in the previous inequality, we obtain that $(g(n)+1)^2 \leq 2n-(g(n)+1)+4$. That is, $\binom{g(n)+2}{2} \leq n+2$. On the other hand, by our definition of $g(n)$ we have that $\binom{g(n)+1}{2} > n$ and that $g(n) \geq 3$, hence $\binom{g(n)+2}{2} = \binom{g(n)+1}{2} + g(n) + 1 \geq n + 4$, a contradiction. ∎

Proposition 24 *Let c be a proper coloring of $D(P)$, and let $C := v_1 v_2 ... v_r v_1$ be a cycle of $\mathcal{P}$ of length $r \geq 3$. If any two distinct edges of C receive distinct color, and each edge e of C is not crossed by any other edge of $\mathcal{P}$ with color $c(e)$, then*

$$\chi(D(P \setminus \{v_1, v_2, ..., v_r\})) \leq |c(D(P))| - r.$$

Proof: Let e be an edge of C with endpoints v_i and v_{i+1}. Since any edge of $\mathcal{P}$ with same color than e has a common endpoint with e, then $\mathcal{P} \setminus \{v_i, v_{i+1}\}$ has no more edges of color $c(e)$. Then $\mathcal{P} \setminus \{v_1, \ldots, v_r\}$ has no edges of color $c(e')$ for any edge e' of C. Therefore,

$$\chi(D(P \setminus \{v_1, v_2, ..., v_r\})) \leq |c(D(P))| - r.$$

∎

3.4 The value of $\chi(D(V_{p,q}))$

Our goal in this section is to determine $\chi(D(P))$ for the case in which P is the double chain $V_{p,q}$.

We recall that for p and q positive integers with $p \geq q$, the *double chain* $V_{p,q}$ is a $(p+q)$-point set, with $a_1, \ldots, a_q$ on the upper line L_1 and $b_1, \ldots, b_p$ on the lower line L_2. Both lines are convex with opposed concavity. Besides, for every i and j the line connecting a_i and a_j leaves all points of L_2 below, and the line connecting b_i and b_j leaves all points of L_1 above. The numbering of the points in both lines is from left to right. See Figure 3.10. For the rest of this section, we shall use A_q and B_p to denote the sets $\{a_1, \ldots, a_q\}$ and $\{b_1, \ldots, b_p\}$, respectively.

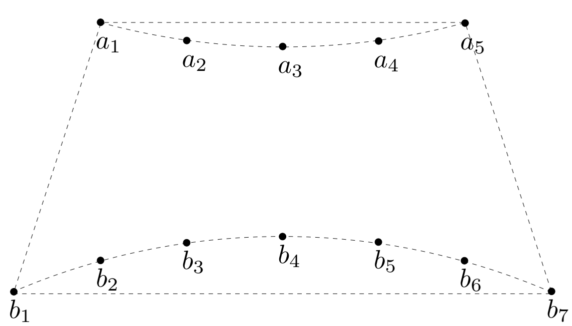

Figure 3.10: $V_{7,5}$.

We will prove that $f(p) + q$ colors are necessary and sufficient for coloring $D(V_{p,q})$.

Proposition 25 *Let c be a proper coloration of $D(V_{p,1})$ and let $i \in \{1, 2, ..., p\}$. If neither a_1 nor b_i is an apex of c and $p \geq 2$, then the chromatic class of c containing the edge a_1b_i is a triangle of the form $a_1b_ib_ja_1$ for some $b_j \in B_p \setminus \{b_i\}$.*

Proof: Let a_1 and b_i as in the statement of the proposition, and let e be the edge with endpoints a_1 and b_i. Because none of a_1 and b_i is apex of c, then e

belongs to a thrackle T. Since e is not crossed by any other edge with endpoints in $V_{p,1}$, then every edge of $T \setminus \{e\}$ must be incident with exactly one of a_1 or b_i and, hence, $|T| \geq 3$.

For v a point of $V_{p,1}$, let $d_T(v)$ be the number of edges of T incident with v. From the location of the points of $V_{p,1}$ and the fact that T is not a star, it is easy to see $d_T(a_1) \leq 2$ and $d_T(b_i) \leq 2$.

Since $|T| \geq 3$ and $d_T(a_1) \leq 2$, then T contains an edge e' with endpoints b_i and b_j for some $j \neq i$ and, hence, e and e' are the only edges of T incident with b_i. This and $|T| \geq 3$ imply that $T \setminus \{e\}$ contains an edge e'' which is incident with a_1. Since no edge of the form $a_1 b_r$ crosses e', then e'' must be incident with b_j. ∎

Lemma 26 *If $p \geq 3$, then $\chi(D(V_{p,1})) = 1 + f(p)$.*

Proof: Observe that if we color $D(V_{p,1})$ by cloring B_p optimally, using $f(p)$ colors, and a_1 being an apex, we will have that $\chi(D(V_{p,1})) \leq 1 + f(p)$. In Figure 3.11 we color $V_{6,1}$ to illustrate this idea.

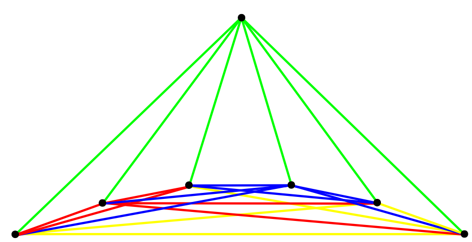

Figure 3.11: Coloring of $D(V_{6,1})$, using $1 + f(6)$ colors, by coloring optimally B_6 and a_1 as an apex. Similarly, for $p \geq 3$, we can color $D(V_{p,1})$ using $1 + f(p)$ colors, by coloring optimally B_p and a_1 as an apex.

Now we prove that $\chi(D(V_{p,1})) \geq 1 + f(p)$. Let c be an optimal coloring of $D(V_{p,1})$. We may assume that c, restricted to B_p uses $f(p)$ colors, as otherwise we are done.

First we show that $\chi(D(V_{3,1})) = 1 + f(3) = 2$. A proper coloring of $D(V_{3,1})$ is given in Figure 3.12, showing that $\chi(D(V_{3,1})) \leq 2$. On the other hand, since the segments $a_1 b_2$ and $b_1 b_3$ are disjoint, then they cannot receive the same color in any proper coloring of $D(V_{3,1})$. This implies that $\chi(D(V_{3,1})) \geq 2$, as required.

Assume that $p \geq 4$. Now we will take $\chi(D(V_{p-1,1})) = 1 + f(p-1)$ as our induction hypothesis.

If a_1 is an apex of c, Proposition 19 implies $|c(D(V_{p,1}))| = |c(D(B_p))| + 1 = f(p) + 1$, as desired. Similarly, if some $b_i \in B_p$ is an apex, Proposition 19 implies

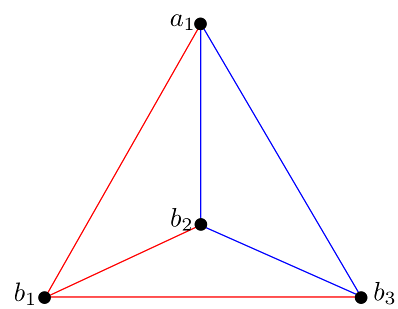

Figure 3.12: A proper coloring of $D(V_{3,1})$, it uses two colors, that is $\chi(D(V_{3,1})) \leq 2$. On the other hand, since a_1b_2 and b_1b_3 are disjoint, they must have different color, that is $\chi(D(V_{3,1})) \geq 2$. Hence $\chi(D(V_{3,1})) = 2$.

$|c(D(V_{p,1}))| = |c(D(V_{p,1} \setminus \{b_i\}))| + 1$. As $V_{p,1} \setminus \{b_i\} = V_{p-1,1}$, then Proposition 19, the induction hypothesis, and Lemma 16 imply that $|c(D(V_{p,1}))| = |c(D(V_{p-1,1}))| + 1 = 1 + f(p-1) + 1 \geq f(p) + 1$, as desired.

Thus we may assume that c has no stars and, hence, c has no apexes. In particular, this implies that there are two distinct edges e and e' such that both are incident with a_1 and $c(e) \neq c(e')$. By applying Proposition 25 to both e and e', we have that the chromatic classes of c containing e and e' are, respectively, two edge disjoint triangles of the form $a_1b_ib_ja_1$ and $a_1b_{i'}b_{j'}a_1$. But this implies that b_ib_j (respectively, $b_{i'}b_{j'}$) is the only edge between points in B_p that has color $c(e)$ (respectively, $c(e')$). See Figure 3.13. This and Proposition 23 imply $|c(D(B_p))| \geq 1 + f(p)$, a contradiction. ∎

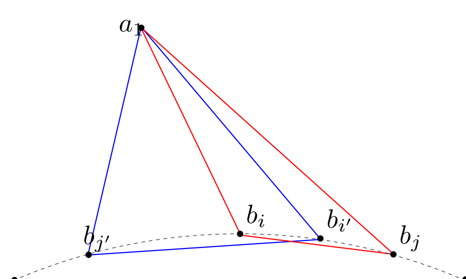

Figure 3.13: If c is an optimal coloration of $D(V_{p,1})$ without stars, then the chromatic class containing any edge incident with a_1 is a triangle of the form $a_1b_rb_{r'}a_1$ for some $b_r, b_{r'} \in B_p$.

Now we can prove the main result of this chapter. We will find the exact number of colors needed to color the edges in $\mathcal{D}(V_{p,q})$.

Theorem 27 *Let p and q be positive integers with $p \geq 3$ and $p \geq q$. Then*

$$\chi(D(V_{p,q})) = q + f(p).$$

Proof: Let $\mathcal{D}$ be the rectilinear drawing of K_{p+q} induced by $V_{p,q}$.

First we show that $\chi(D(V_{p,q})) \leq q + f(p)$: color the edges of B_p with $f(p)$ colors [17]. For each of the q vertices in A_q, color the edges incident to them, that have not been colored yet, with a new color. This yields a proper coloring of $D(V_{p,q})$ with $q + f(p)$ colors. See Figure 3.14.

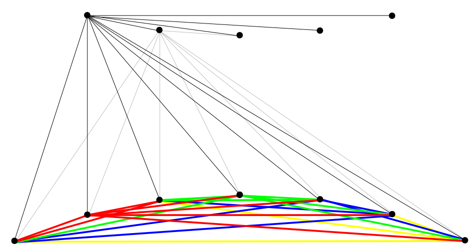

Figure 3.14: We will color $D(V_{7,5})$ by coloring optimally B_7 and every element in A_5 being an apex. At the end, we will use at most $5 + f(7)$ colors. We generalize this idea to color $D(V_{p,q})$ using at most $q + f(p)$ colors.

In order to apply induction, we prove that $\chi(c(D(V_{3,2}))) = 3$: observe that if we color optimally B_3 and a_1 and a_2 being apexes, we color $D(V_{3,2})$ with at most three different colors. See Figure 3.15 left. On the other hand, assume that $c(a_1b_1) = blue$ and $c(a_2b_3) = red$. See Figure 3.15 right. Lets consider $c(a_1a_2)$. We have three cases:

- $c(a_1a_2)$ is different from *blue* and *red*. Clearly $\chi(D(V_{3,2})) \geq 3$.
- $c(a_1a_2) = blue$. We must have that $c(b_1b_2)$ cannot be *blue* nor *red*. Hence we must use a different color, yielding that $\chi(D(V_{3,2})) \geq 3$.
- $c(a_1a_2) = red$. Now $c(b_2b_3)$ cannot be *blue* nor *red*. Hence we must use a different color, again, yielding that $\chi(c(D(V_{3,2}))) \geq 3$.

Now we show that $\chi(D(V_{p,q})) \geq q + f(p)$. From Lemma 26 the theorem holds when $q = 1$ and $p \geq 3$. Now we assume that $q \geq 2$ and $p \geq 4$.

Seeking a contradiction, suppose that $V_{p,q}$ is a counterexample of minimum order. That is, suppose that $V_{p,q}$ is the set such that $p + q$ is minimum and has an optimal coloration c which $|c(D(V_{p,q}))| = \chi(D(V_{p,q})) < q + f(p)$, with $q \geq 2$ and $p \geq 4$.

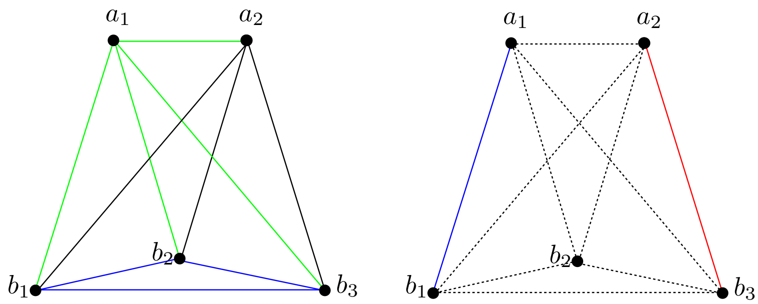

Figure 3.15: On the left, we colored optimally B_3 and a_1 and a_2 as stars to show that $c(D(V_{3,2})) \leq 3$. On the right we colored the edges a_1b_1 and a_2b_3 to show that $c(D(V_{3,2}))$ uses three different colors.

Case 1. c has a star with apex s. If $q = p$, then by interchanging the roles of A_q and B_p if necessary, we may assume without loss of generality that $s \in A_q$. So, $V_{p,q-1} = V_{p,q} \setminus \{s\}$, and by Proposition 19 we have that $\chi(D(V_{p,q-1})) = \chi(D(V_{p,q})) - 1 < q + f(p) - 1$, contradicting the minimality of $V_{p,q}$.

Similarly, if $s \in B_p$, then $q < p$ (by our previous assumption) and $V_{p-1,q} = V_{p,q} \setminus \{s\}$. Then by Proposition 19 we get $\chi(D(V_{p-1,q})) = \chi(D(V_{p,q})) - 1 < q + f(p) - 1$. Since $\chi(D(V_{p-1,q})) = q + f(p-1)$, then $q + f(p-1) < q + f(p) - 1$, which contradicts Lemma 16.

Case 2. c has no stars. Let $E := \{a_1b_1, b_1b_p, b_pa_q, a_qa_1\}$ be the set of edges in $\operatorname{conv}(V_{p,q})$, and γ be the number of chromatic classes of c restricted to E. Clearly $\gamma \in \{2, 3, 4\}$. It is easy to see that if $\gamma = 2$, then at least one of such two chromatic classes is a star of c. See Figure 3.16. We may assume that $\gamma \in \{3, 4\}$.

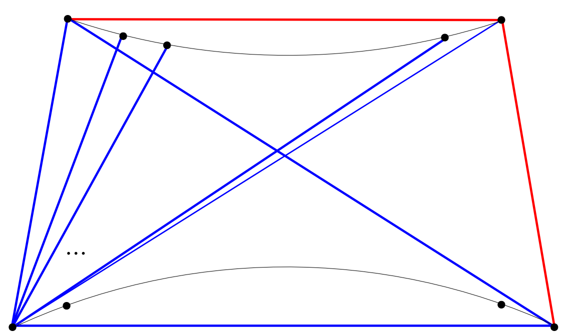

Figure 3.16: If $c(a_1b_1) = c(b_1b_p)$ and $c(a_1a_q) = c(a_qb_p)$, then $c(a_1b_p)$ makes b_1 or a_q (or both) an apex. Here a_q is an apex, hence we apply induction on $V_{q-1,p}$.

- $\gamma = 4$. Then $\chi(D(V_{p,q} \setminus \{a_1, b_1, a_q, b_p\})) < q + f(p) - 4$, by Proposition 24 and the definition of c. Since $V_{p-2,q-2} = V_{p,q} \setminus \{a_1, b_1, a_q, b_p\}$, see Figure 3.17, and $\chi(D(V_{q-2,p-2})) = q - 2 + f(p-2)$, then $q - 2 + f(p-2) < q + f(p) - 4$, which contradicts Lemma 16.

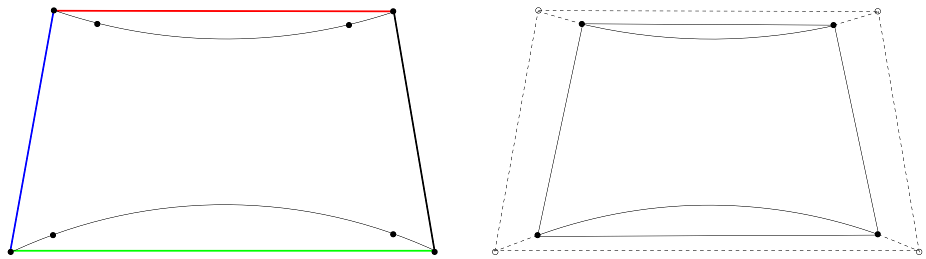

Figure 3.17: When the edges of $\mathrm{conv}(V_{p,q})$ are colored with four different colors, we can remove the vertices in $\mathrm{conv}(V_{p,q})$ to eliminate four chromatic classes.

- $\gamma = 3$. Then exactly two edges in $\mathrm{conv}(V_{p,q})$ are of the same color; moreover these edges share an endpoint. Without loss of generality assume that their common endpoint is a_1. Assume that a_1a_q and a_1b_1 are colored *blue*. Since all the chromatic classes in c are thrackles, then the edge b_1a_q must also be colored *blue*. Note that any other *blue* edge must be incident to a_1 and its other endpoint must be in B_q. Now we recolor *blue* all the edges incident with a_1 and having the other endpoint in B_q. See Figure 3.18.

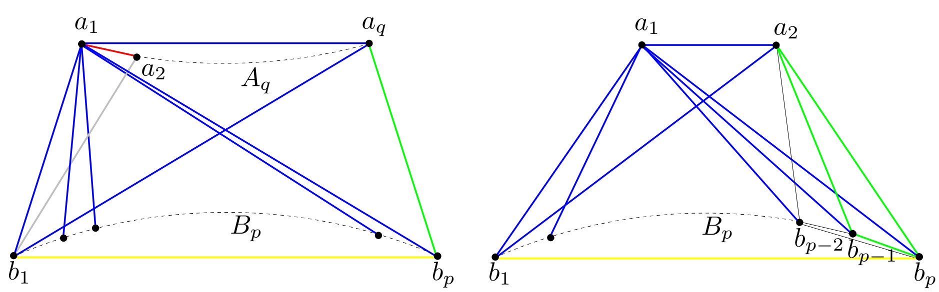

Figure 3.18: Here we illustrate the only two (up to symmetry) possibilities for the case $\gamma = 3$. On the left we have the case in which $q \geq 3$. On the right we have the case in which $q = 2$.

First let us assume that $q \geq 3$. The proof for the case when the edges in $\mathrm{conv}(V_{p,q})$ having same color sharing the one vertex in B_q is totally analogous. Lets consider a_2. We have that the edge a_1a_2 does not cross any other edge, and in particular a_1a_2 cannot be *blue*. Suppose that a_1a_2 is *red*. If b_1a_2 is also colored *red*, then the *red* chromatic class is a star, a contradiction. Thus b_1a_2 is not *red*. Since b_1a_2 cannot be colored *blue*, we assume that it is colored *gray*. See Figure 3.18 (left). Since b_1a_2 is crossed only by *blue* edges, then any other *gray* edge must be incident to b_1 or a_2. Also note that every *red* edge must be incident to a_1 or a_2. These observations together imply that c when restricted to $V_{p,q} \setminus \{a_1, a_2, b_1\}$ is a coloring of $D(V_{p,q} \setminus \{a_1, a_2, b_1\})$ with less than $q + f(p) - 3$ colors. Observe that $V_{p,q} \setminus \{a_1, a_2, b_1\} = V_{q-2,p-1}$. By Proposition 16, $q + f(p) - 3 \leq (q-2) + f(p-1)$; this contradicts the minimality of $p+q$ and also contradicts Lemma 16.

Now suppose that $q = 2$. Then $A_q = \{a_1, a_2\}$. By symmetry, we may assume that color on the edges of $\text{conv}(V_{p,q})$ are placed as in Figure 3.18 (right), and that a_2b_p is *green*. Lets consider b_{p-1}. Observe that $b_{p-1}b_p$ does not cross any other edge, and any edge crossing a_2b_{p-1} is *blue*. Also note that a_2b_{p-1} cannot be *blue*. If a_2b_{p-1} and $b_{p-1}b_p$ receive the same color, different from *green*, then the chromatic class containing them must be a star. Similarly, if $b_{p-1}a_2, b_{p-1}b_p$ and a_2b_p receive distinct colors, then we can proceed as in previous paragraph and deduce that $V_{p-2,q-1} = V_{p,q} \setminus \{a_2, b_{p-1}, b_p\}$ is a counterexample that contradicts the minimality of $p + q$.

Thus we may assume that at least one of $b_{p-1}a_2$ or $b_{p-1}b_p$ is *green*. We claim that both are *green*. Because a_2b_p is not crossed by any edge, then any other *green* edge must be adjacent to exactly one of a_2 or b_p. This and the fact that the *green* chromatic class is not a star, imply that for each $v \in \{a_2, b_p\}$ there exists at least one *green* edge distinct of a_2b_p which is incident with v. Let a_2x and b_py be any couple of such *green* edges. Clearly, $x, y \in B_p \setminus \{b_p\}$. Since the *green* edges incident with a_2 are crossed only by *blue* edges, then we must have that $x = y$. This and the fact that at least one of a_2b_{p-1} or $b_{p-1}b_p$ is *green* imply that $b_{p-1} = x = y$. This implies that the *green* chromatic class consists precisely of $a_2b_{p-1}, b_{p-1}b_p$ and a_2b_p.

Lets consider b_{p-2}. Note that $b_{p-1}b_{p-2}$ does not cross any other edge, and that any edge crossing a_2b_{p-2} is *blue*. Also note that none of a_2b_{p-2} and $b_{p-2}b_p$ can be *blue* or *green*. Again, if a_2b_{p-2} and $b_{p-2}b_p$ receive the same color, then the chromatic class containing them must be a star. Thus we assume that a_2b_{p-2} and $b_{p-2}b_p$ have distinct colors. This implies that the color of at least one of a_2b_{p-2} or $b_{p-2}b_p$ is different from the color of $b_{p-1}b_{p-2}$. Let $v \in \{a_2, b_p\}$ such that $c(b_{p-1}b_{p-2}) \neq c(b_{p-2}v)$. Since none of $b_{p-1}b_{p-2}$ and $b_{p-2}v$ can be *green*, then the colors of $b_{p-1}b_{p-2}, b_{p-1}v$, and $b_{p-2}v$ are distinct. From this and the fact that any edge crossing $b_{p-2}v$ is *blue* or incident with b_{p-1}, it follows that $V_{p,2} \setminus \{v, b_{p-1}, b_{p-2}\}$ is a counterexample that contradicts the minimality of $p + q$. The result follows.

Hence $\chi(D(V_{p,q})) \geq f(p) + q$. ∎

Observe that if $p = q = \frac{n}{2}$, then

$$\chi(D(V_{n/2,n/2})) = \frac{n}{2} + f\left(\frac{n}{2}\right) = n - \left\lfloor \sqrt{n+1/4} - 1/2 \right\rfloor > f(n).$$

Chapter 4

Convex Decompositions

Theorem 1 take us to the following question: given a positive integer $k \geq 4$, there exists an integer n_k such that in any point set with at most n_k points on the plane contains a convex k–gon?

This problem was solved in the affirmative by P. Erdős and G. Szekeres [6] with the Erdős–Szekeres' Theorem:

Theorem 28 *For every positive integer k there exist an integer n_k such that if $n \geq n_k$, any n–point set contains a convex k–gon.*

This theorem has been studied because of its beauty and because its a great challenge to find the exact value of n_k. More than 60 years passes without significant changes.

So far, what we know about n_k is that:

$$2^k + 1 \leq n_k \leq \binom{2k-5}{k-2} + 2.$$

The lower bound is obtained in the same paper [6], and the upper bound is obtained by G. Toth and P. Valtr [20]. It is important to note that the polygon obtained can contain elements of the point set in its interior.

If a polygon does not contain any element of the point set in its interior, we say that such polygon is *empty*.

A well known question is the following: given an integer k, there exists an integer h_k such that, in any point set with at most h_k elements is possible to obtain an empty k–gon?

Trivially we have that $h_3 = 3$. In Figure 4.1 (left) we can see a 4-point set without a convex quadrilateral, so $h_4 \geq 5$, and by Esther Klein's result $h_4 = 5$. In Figure 4.1 (right) we can see a 9–point set without convex pentagons, so $h_5 \geq 10$. Harborth proved in [13] that $h_5 = 10$.

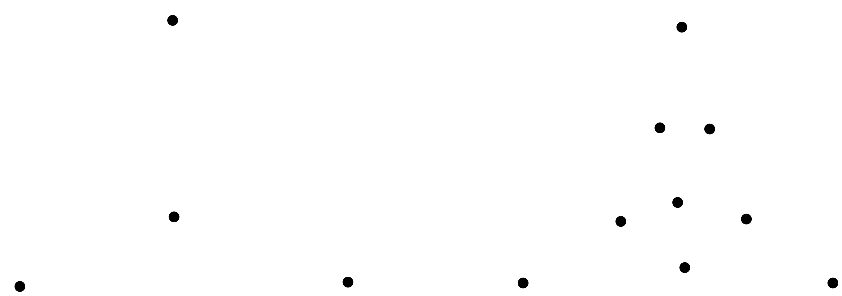

Figure 4.1: A point set with 4 elements without convex quadrilateral and a point set with 9 elements without convex pentagon.

J.D. Horton [14] gives point sets arbitrarily large in which there are no empty heptagons, and hence without empty k–gons for $k \geq 7$, showing that h_k does not exist, for $k \geq 7$.

After Horton's result, only remains to find the exact value for h_6. This problem remained open for a long time. M. Overmars [19] try to show that h_6 do not exist by constructing point sets arbitrarily large without empty convex hexagons, but the largest set he found was a 29-point set. Finally T. Gerken [12] proved that h_6 *do* exist. We only know that $30 \leq h_6 \leq n_9$.

In this work we are not interested in the number of vertices of the polygons. We are interested in shattering conv(P) with the smallest number of empty convex polygons with its vertices in P. Here, polygons are considered as its border together its interior.

Formally, a *convex decomposition* Γ of P, is a family of convex polygons $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_m\}$ satisfying the following:

($\mathcal{CD}$1) The vertices of every γ_i are in P, $i = 1, 2, ..., m$.

($\mathcal{CD}$2) Every γ_i is empty, $i = 1, 2, ..., m$.

($\mathcal{CD}$3) γ_i and γ_j are interior–disjoint $1 \leq i \neq j \leq m$.

($\mathcal{CD}$4) The union $\gamma_1 \cup \gamma_2 \cup ... \cup \gamma_m$ is conv(P).

Just as a small observation: we can state the Rectilinear Crossing Number as finding a point set P which minimizes the elements of a family of convex quadrilaterals only satisfying ($\mathcal{CD}1$). Since we allow elements of P be in the interior of the quadrilaterals and that they can share their interiors, the family not necessarily satisfies ($\mathcal{CD}2$), ($\mathcal{CD}3$) and ($\mathcal{CD}4$).

4.1 Convex decompositions starting from a triangulation

Observe that a triangulation of P is a convex decomposition with exactly $2n - h(P) - 2$ elements. We are interested in obtaining a convex decomposition which improves such number.

Let $cd(P) = \min\{|\Gamma| : \Gamma \text{ is a convex decomposition of } P\}$, and let $cd(n)$ be the maximum value of cd among all the n point sets in the plane. Urrutia [21] conjectured that $cd(n) \leq n+1$, but this was disproved by Aichholzer and Krasser [1] by showing that $cd(n) \geq n+2$. Later, García-López and Nicolás [11] gave a n–point set such that $cd(n) \geq \frac{11n}{10}$.

Neumann, Rivera and Urrutia [18] show that, for any given n-point set there exists a convex decomposition with at most $\frac{10n-18}{7}$ elements. This bound was improved by Hosono, [15] by showing that $cd(n) \leq \left\lceil \frac{7}{5}(n-3) \right\rceil + 1$.

The previous bounds are existence theorems. Here we show how to obtain a specific convex decomposition that reaches the $\frac{10n}{7}$ bound.

4.2 Convex Decompositions with $\frac{10n}{7}$ elements

For the rest of the section we use the following labeling: p_1 will be the element in P with lowest y coordinate (and lowest x coordinate in case of a tie), and every $p \in P \setminus \{p_1\}$, will be labelled as $p_2, p_3, \ldots, p_n$ in their angular order around p_1, see Figure 4.2 (left).

Now we label the elements of the polygonal $p_3 p_4 \ldots p_{n-1}$ as follows: if $\text{conv}(\{p_1, p_{i-1}, p_i, p_{i+1}\})$ is a triangle, then we say p_i is *negative*, labeled $-$. Otherwise we say p_i is *positive*, labeled $+$. See Figure 4.2 (right).

We recall that Graham triangulation $\Gamma_G(P)$ of P is obtained recursively as follows:

Once that $\Gamma_G(\{p_1, \ldots, p_{k-1}\})$ has been costructed, we construct $\Gamma_G(\{p_1, \ldots,$

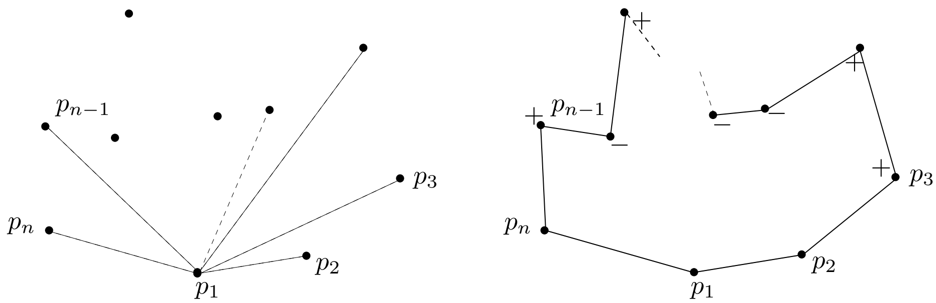

Figure 4.2: Labeling elements of P.

$p_{k-1}, p_k\}$) by drawing the line segments $p_k v$, for all vertices v of conv($\{p_1, \ldots,$ $p_{k-1}\}$), that do not cross any other segment of $\Gamma_G(\{p_1, \ldots, p_{k-1}, p_k\})$. See Figure 4.3.

Observe that, at the end of the procedure, $\Gamma_G(P)$ will have all edges $p_1 p_i$, with $i = 2, 3, ..., n$ and all the edges of the form $p_j p_{j+1}$, with $j = 2, 3, ..., n-1$. See Figure 4.3.

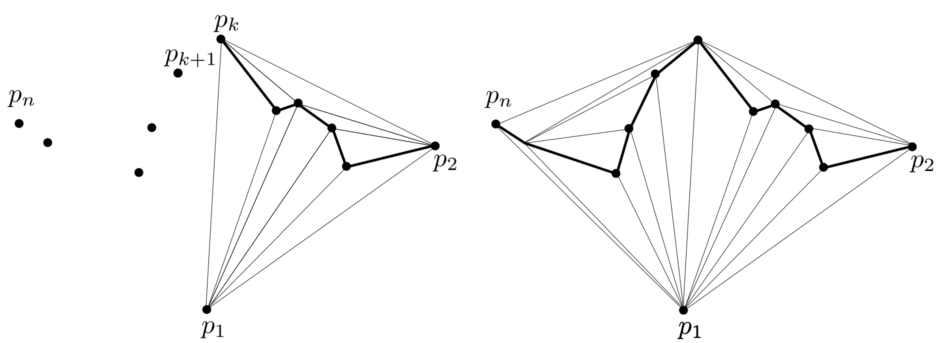

Figure 4.3: Application of Graham's algorithm to P. Observe that, at the end, it will get all edges $p_1 p_i$ and $p_j p_{j+1}$, for $i = 2, 3, ..., n$ and $j = 2, 3, ..., n-1$.

4.2.1 Using Graham's triangulation.

We assume that P is now triangulated with Graham's algorithm. Let A and B be the subsets of P containing all positive and negative elements respectively. Notice that $A \cup B = P \setminus \{p_1, p_2, p_n\}$.

Observation 29 *If P is the set of vertices of a convex polygon, then $cd(P) = 1$. So, we will assume that all point sets under consideration are different from the point set in convex position. Then we can assume that $B \neq \emptyset$.*

We partition A into its maximal subsets of consecutive positive points, labeled $A_1, \ldots, A_k$ in the angular order around p_1. See Figure 4.4 (left). If $p_3 \notin A$, then we make $A_1 = \emptyset$, similarly, if $p_{n-1} \notin A$, then we make $A_k = \emptyset$. Also, we divide B into its maximal subsets of consecutive points. We label them as $B_1, \ldots, B_{k-1}$, in the angular order around p_1. See Figure 4.4 (left). That is, $A = A_1 \cup A_2 \cup ... \cup A_k$ and $B = B_1 \cup B_2 \cup ... \cup B_{k-1}$. Observe that $k \leq \frac{n}{2}$.

In [16] it was shown that for a given triangulation T of P, and a given pair of vertices $u, v \in P$, we can modify T to get a triangulation T' containing the edge uv, by adding $\overline{uv}$, removing the edges of T intersecting it, and triangulating separately the resulting two polygons γ and γ' sharing the edge uv. We will modify Γ_G as follows:

(C1) For $j = 1, 2, \ldots, k-1$, suppose that $B_j = \{p_{i_j}, p_{i_j+1}, \ldots, p_{i_j+r_j-1}\}$. Note that $r_j > 0$ in each case. Now we modify T_G, if necessary, by adding the edge $e_j := p_{i_j-1}p_{i_j+r_j}$ as described in previous paragraph. Let B'_j be the convex polygon with vertex set $\{p_{i_j}, p_{i_j+1}, \ldots, p_{i_j+r_j-1}, p_{i_j-1}, p_{i_j+r_j}\}$. Then we remove any diagonal inside of B'_j. Let $\mathcal{B} = B'_1 \cup \cdots \cup B'_{k-1}$. See Figure 4.4 (right).

(C2) Also, for every $j = 1, 2, \ldots, k$, if $A_j = \{p_{i_j}, p_{i_j+1}, \ldots, p_{i_j+r_j-1}\} \neq \emptyset$, then we remove the edges p_1p, for all $p \in A_j$, to obtain the empty convex polygon defined by $A_j \cup \{p_1, p_{i_j-1}, p_{i_j+r_j}\}$. If $A_1 = \emptyset$, then we make $A'_1 = \{p_1, p_2, p_3\}$, similarly, if $A_k = \emptyset$, then we make $A'_k = \{p_1, p_{n-1}, p_n\}$. Let $\mathcal{A} = A'_1 \cup \cdots \cup A'_k$. See Figure 4.4 (right).

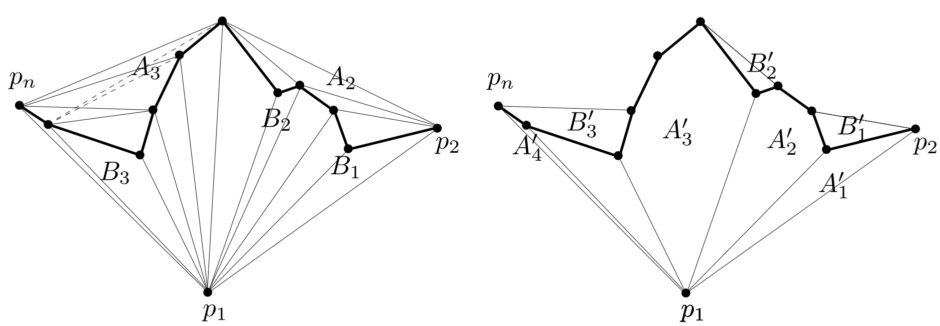

Figure 4.4: In this example, $k = 4$ and $A_1 = A_k = \emptyset$. After modifying Graham's triangulation, according to C1 and C2, we obtain A'_1, B'_1, A'_2, B'_2, A'_3, B'_3 and A'_4.

Let Γ be the convex decomposition obtained after applying (C1) and (C2) to $\Gamma_G(P)$, and $\mathcal{T}_B$ be the set of triangles in Γ with vertex set $\{p_1, p_m, p_{m+1}\}$, for all pair p_m, p_{m+1} of negative points. See Figure 4.5 (left). Observe that $|\mathcal{T}_B| = \sum_{i=1}^{k-1}(|B_i| - 1)$.

Let $\mathcal{T}$ be the set of triangles contained in conv(P), which are disjoint from $\mathcal{A} \cup \mathcal{B} \cup \mathcal{T}_B$. See Figure 4.5 (right).

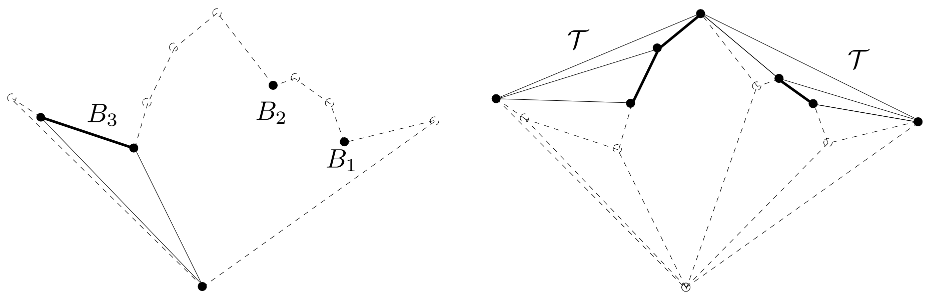

Figure 4.5: Here $\mathcal{T}_B = \{\triangle p_1p_{10}p_{11}\}$ and $\mathcal{T} = \{\triangle p_2p_4p_5, \triangle p_2p_5p_7, \triangle p_7p_8p_{12}, \triangle p_8p_9p_{12}\}$.

Proposition 30 $|\mathcal{T}| = n - |B| - h(P)$.

Proof: Observe that every A'_i and B'_j contains $|A_i| + 1$ and $|B_j|$ triangles of Γ' respectively. Also, every B'_i is adjacent to $|B_i| - 1$ triangles in $\mathcal{T}_B$. We must have that

$$2n - h(P) - 2 = \sum_{i=1}^{k}(|A_i| + 1) + \sum_{i=1}^{k-1}|B_i| + \sum_{i=1}^{k-1}(|B_i| - 1) + |\mathcal{T}|. \tag{4.1}$$

Since $|A| = \sum_{i=1}^{k}|A_i|$, $|B| = \sum_{i=1}^{k-1}|B_i|$ and $|A| + |B| = n - 3$, then equation (4.1) will be

$$2n - h(P) - 2 = |A| + k + |B| + |B| - (k-1) + |\mathcal{T}| = n - 2 + |B| + |\mathcal{T}|.$$

Hence $|\mathcal{T}| = n - |B| - h(P)$, as claimed.

∎

Lemma 31 $|\Gamma| = n + k - h(P)$.

Proof: Since $\Gamma = \mathcal{A} \cup \mathcal{B} \cup \mathcal{T}_B \cup \mathcal{T}$, then

$$|\Gamma| = k + (k-1) + |\mathcal{T}_B| + |\mathcal{T}| = n + k - h(P). \tag{4.2}$$

See Figure 4.6. ∎

Since $k \leq \frac{n}{2}$, then $|\Gamma| \leq \frac{3n}{2} - h(P)$ [18]. And, if $k \leq \frac{3n}{7}$, then we prove our main result.

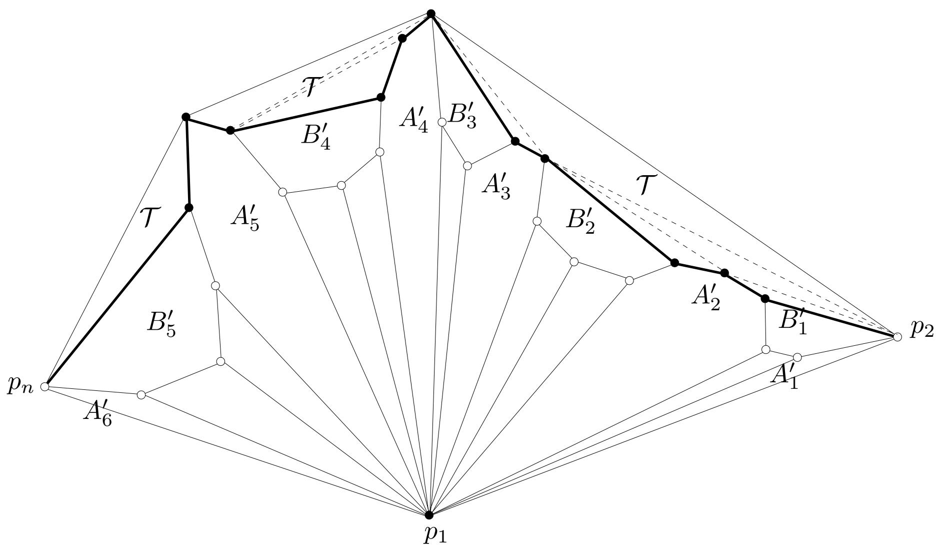

Figure 4.6: A point set P with 27 elements, and Γ as described above. Here $k = 6$, $h(P) = 5$ and $A_1 = A_k = \emptyset$. Then $|\Gamma| = n + k - h(P) = 28$.

4.2.2 Obtaining a convex decomposition for $\frac{3n}{7} < k \leq \frac{n}{2}$

Now assume that $\frac{3n}{7} < k \leq \frac{n}{2}$. We use the following notation: if the signs of p_i and p_{i+1} are different, for any $3 \leq i \leq n-2$, then we say that P is a $\pm$*–set*.

Lemma 32 *Suppose that $n = 6m + 2$, for a positive integer m. If P is a $\pm$ set and p_3 has label $-$, then P has a convex decomposition Γ with $\frac{4n}{3} - h(P) + 1$ elements.*

Proof: We will use the following notation: if two given polygons α and β, of a convex decomposition, share an edge e, we denote $\alpha \overline{\cup} \beta$ the operation of combining α and β, that is, we let $\alpha \overline{\cup} \beta := (\alpha \cup \beta) - e$. We remark that $\alpha \overline{\cup} \beta$ is a polygon not necessarily convex.

We define the subsets $Q_i = \{p_1, p_i, p_{i+1}, ..., p_{i+6}\}$, for $i = 2, 8, \ldots, n-6$. We will obtain a convex decomposition Γ_i for every Q_i and use it to get a convex decomposition of P. Observe that, since in every Q_i there are three negative points, then $\text{conv}(Q_i)$ has 3, 4 or 5 vertices.

We analyze only the case when $i = 2$, that is, $Q_2 = \{p_1, p_2, ..., p_8\}$. The analysis for the remaining Q_i sets can be handled analogously. We divide in cases according on the number of vertices in $\text{conv}(Q_2)$.

Let $\Delta_1 := \triangle p_1p_7p_8$, $\Delta_2 := \triangle p_6p_7p_8$, $S := \square p_1p_5p_6p_7$, and T be the interior of the triangle $\triangle p_1p_2p_4$. See Figure 4.7.

Case 1. Suppose that $\text{conv}(Q_2)$ is the pentagon $p_1p_2p_4p_6p_8$. Lets consider the segment $\overline{p_2p_5}$ and the line ℓ containing p_5 and p_6, directed from p_5 to p_6. Make $\mathcal{L}$ and $\mathcal{U}$ the left and right open half plane defined by ℓ respectively. Now we divide T.

If $p_2 \in \mathcal{U}$, then we make T_1 the region $\mathcal{L} \cap T$, T_2 the region in T contained in the angle formed by $\overline{p_2p_5}$ and ℓ. T_3 will be the region $T - (T_1 \cup T_2)$. See Figure 4.7 (left).

On the other hand, if $p_2 \in \mathcal{L}$, then T_3 will be the region $\mathcal{U} \cap T$, T_2 the region of T contained in the angle formed by $\overline{p_2p_5}$ and ℓ. T_1 will be the region $T - (T_3 \cup T_2)$. See Figure 4.7 (right).

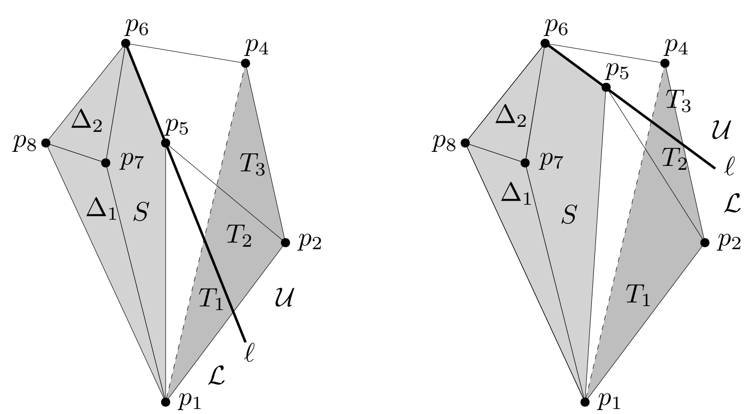

Figure 4.7: Case when $\text{conv}(Q_2)$ is a pentagon. We define the regions T_1, T_2 and T_3. We will use $\Delta_1 := \triangle p_1p_7p_8$, $\Delta_2 := \triangle p_6p_7p_8$, and $S := \square p_1p_5p_6p_7$.

Case 1.1. Suppose that $p_3 \in T_1$. We make $\alpha := S \overline{\cup} \triangle p_1p_3p_5$ and $\beta := \square p_2p_4p_5p_3$ to get a convex decomposition Γ_2 of Q_2 with six elements $\Gamma_2 := \{\Delta_1, \Delta_2, \triangle p_1p_2p_3, \triangle p_4p_5p_6, \alpha, \beta\}$.

Observe that the pentagon and the quadrilateral α and β are convex since

$p_3 \in \mathcal{L}$, and p_3 and p_4 are on opposite sides with respect to $\overline{p_2p_5}$.

Case 1.2. Suppose that $p_3 \in T_2$, and hence ℓ leaves p_2 and p_3 in the same half plane. If p_2 and p_3 are in $\mathcal{U}$, then we make α the pentagon with vertex set $\{p_2, p_4, p_6, p_5, p_3\}$, and so $\Gamma_2 = \{\Delta_1, \Delta_2, \triangle p_1p_2p_3, \triangle p_1p_3p_5, S, \alpha\}$ is the required.

If p_2 and p_3 are in $\mathcal{L}$, we make β the hexagon $S\overline{\cup}\square p_1p_2p_3p_5$, and so $\Gamma_2 = \{\Delta_1, \Delta_2, \triangle p_2p_3p_4, \triangle p_3p_4p_5, \triangle p_4p_5p_6, \beta\}$.

Case 1.3. Suppose that $p_3 \in T_3$. In this case $\alpha := \square p_1p_2p_3p_5$ is always convex. We let $\beta := \square p_5p_3p_4p_6$ and so $\Gamma_2 := \{\Delta_1, \Delta_2, S, \alpha, \beta, \Delta p_2p_3p_4\}$ is the required.

Case 2. conv(Q_2) is a quadrilateral. Then conv$(Q_2) = \square p_1p_2p_6p_8$ or conv$(Q_2) = \square p_1p_2p_4p_8$. By symmetry, we only analyze when conv$(Q_2) = \square p_1p_2p_4p_8$. We proceed as in Case 1. Roughly speaking, we will add $\triangle p_4p_6p_8$ to the previous sets Γ_2:

Let ℓ, and the regions $\mathcal{L}, \mathcal{U}, T_1, T_2, T_3$ as in Case 1. Also let $\Delta_3 := \triangle p_4p_6p_8$. See Figure 4.8.

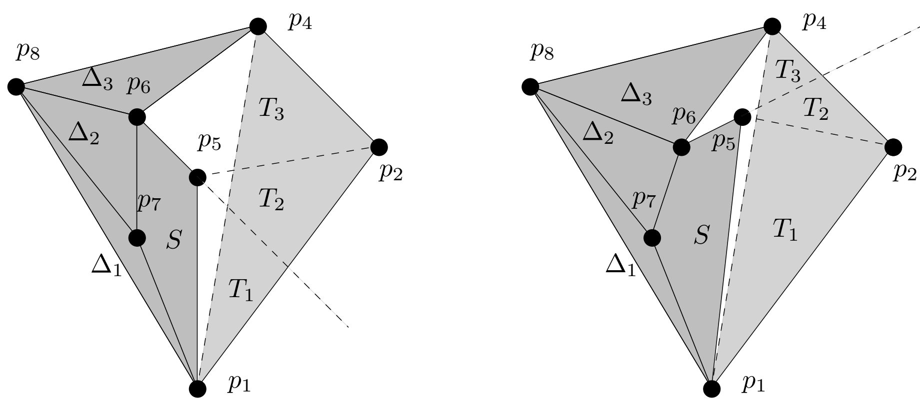

Figure 4.8: Case when conv(Q_2) is the quadrilateral $\square p_1p_2p_4p_8$. We define the regions R_1, R_2 and R_3. Also we will use $\Delta_3 := \triangle p_4p_6p_8$.

Case 2.1. Suppose that $p_3 \in T_1$. We make $\alpha := S\overline{\cup}\triangle p_1p_3p_5$ and $\beta := \square p_2p_4p_5p_3$. Note that $\Gamma_2 := \{\Delta_1, \Delta_2, \Delta_3, \triangle p_1p_2p_3, \triangle p_4p_5p_6, \alpha, \beta\}$ is the required convex decomposition.

Observe that the pentagon and the quadrilateral α and β are convex since $p_3 \in \mathcal{L}$, and p_3 and p_4 are on opposite sides on $\overline{p_2p_5}$.

Case 2.2. Suppose that $p_3 \in T_2$, and hence ℓ leaves p_2 and p_3 in the same half plane. If p_2 and p_3 are in $\mathcal{U}$, then we make α the pentagon with vertex set $\{p_2, p_4, p_6, p_5, p_3\}$, and so $\Gamma_2 = \{\Delta_1, \Delta_2, \Delta_3, \triangle p_1p_2p_3, \triangle p_1p_3p_5, S, \alpha\}$ is the required.

If p_2 and p_3 are in $\mathcal{L}$ we make β the hexagon $S\overline{\cup}\square p_1p_2p_3p_5$, and so $\Gamma_2 = \{\Delta_1, \Delta_2, \Delta_3, \triangle p_2p_3p_4, \triangle p_3p_4p_5, \triangle p_4p_5p_6, \beta\}$.

Case 2.3. Suppose that $p_3 \in T_3$. In this case, again, $\alpha := \square p_1p_2p_3p_5$ is always convex. We let $\beta := \square p_5p_3p_4p_6$ and so $\Gamma_2 := \{\Delta_1, \Delta_2, \Delta_3, \triangle p_2p_3p_4, S, \alpha, \beta\}$ is the required.

Case 3. Suppose that $\text{conv}(Q_2)$ is a triangle. Indeed, suppose that $\text{conv}(Q_2) = \triangle p_1p_2p_8$. If $\square p_2p_4p_6p_8$ is convex, then we just make $\Gamma_2 = \{\triangle p_1p_2p_3, \triangle p_2p_3p_4, \triangle p_4p_5p_6, \Delta_1, \Delta_2, S, \square p_1p_3p_4p_5, \square p_2p_4p_6p_8\}$. See Figure 4.9 (left).

If $\square p_2p_4p_6p_8$ is not convex, then by simmetry we can assume that p_4 is in the interior of $\triangle p_2p_6p_8$.

Let ℓ and ℓ' be the rays directed from p_8 to p_4 and p_8 to p_6 respectively. Now we divide T as follows: R_1 will be the region of T on the right side of ℓ', R_3 will be the region of T on the left side of ℓ, and R_2 will be the interior of the region $T - (R_1 \cup R_3)$. See Figure 4.9 (right).

Similarly as in previous cases, we make a case analysis on the location of p_3.

Case 3.1. Suppose that $p_3 \in R_1$. Also suppose that p_5 is in interior of $\square p_2p_4p_6p_3$. We have that the pentagon $\alpha := p_1p_3p_5p_6p_7$ will be convex.

If p_5 is in the left side of ℓ', then $\beta := \square p_8p_6p_5p_4$ will be convex, so we make $\Gamma_2 = \{\Delta_1, \Delta_2, \triangle p_1p_2p_3, \triangle p_2p_4p_8, \triangle p_3p_2p_4, \triangle p_3p_4p_5, \alpha, \beta\}$.

If p_5 is on the right side of ℓ', then $\gamma := \square p_2p_4p_6p_5$ will be convex. So Γ_2 will be $\{\Delta_1, \Delta_2, \Delta_3, \triangle p_1p_2p_3, \triangle p_2p_4p_8, \triangle p_2p_5p_3, \alpha, \gamma\}$.

Now suppose that p_5 is not in interior of $\square p_2p_4p_6p_3$. Let α be the pentagon $p_2p_4p_6p_5p_3$. If α is convex, then $\Gamma_2 := \{\Delta_1, \Delta_2, \Delta_3, \triangle p_1p_2p_3, \triangle p_1p_3p_5, \triangle p_2p_4p_8, S, \alpha\}$. If α is not convex, then the quadrilateral $\beta := \square p_1p_2p_3p_5$ will be. Then $\Gamma_2 := \{\Delta_1, \Delta_2, \triangle p_2p_4p_8, \triangle p_2p_4p_3, \triangle p_3p_5p_6, S, \beta, \square p_3p_4p_8p_6, \}$.

Case 3.2. Suppose that $p_3 \in R_2$. Let $\alpha := p_3p_2p_4p_6p_5$. If α is convex, then Γ_2 will be the set $\{\Delta_1, \Delta_2, \Delta_3, \triangle p_2p_4p_8, \triangle p_1p_2p_3, \triangle p_1p_3p_5, S, \alpha\}$.

On the other hand, if α is not convex, then Γ_2 will be the set $\{\Delta_1, \Delta_2, \triangle p_2p_8p_4, \triangle p_3p_6p_5, \triangle p_3p_2p_4, \square p_1p_2p_3p_5, \square p_3p_4p_8p_6, S\}$.

And finally.

Case 3.3. Suppose that $p_3 \in R_3$. Let $\gamma := \Box p_8p_4p_3p_2$ and simply make $\Gamma_2 := \{\Delta_1, \Delta_2, \Delta_3, \triangle p_1p_2p_3, \triangle p_4p_6p_5, \gamma, S, \Box p_1p_3p_4p_5\}$.

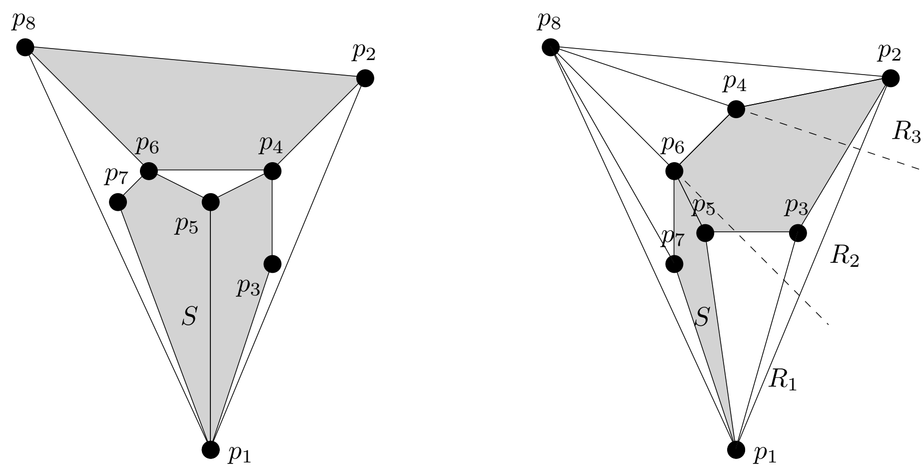

Figure 4.9: Case when $\mathrm{conv}(Q_2)$ is the triangle $\triangle p_1p_2p_8$. We divide in two cases according to $\Box p_2p_4p_6p_8$ if it is convex or not.

Now, for $j = 3, 4, 5$, let C_j be the set $C_j = \{Q_i : \mathrm{conv}(Q_i) \text{ is a } j-\text{gon}\}$. To obtain the desired convex decomposition Γ of P we proceed as follows:

(D1) For every Q_i, with $i = 2, 8, \ldots, n-6$ obtain Γ_i as described in previous cases.

(D2) Triangulate the region(s) of $\mathrm{conv}(P)$ disjoint from $\Gamma_2 \cup \Gamma_8 \cup \cdots \cup \Gamma_{n-6}$, and let $\mathcal{R}$ be the set of such triangles.

Observation 33 *For $i = 8, 14, ..., n-6$, the edges $p_{i-1}p_i$ and p_ip_{i+1} are always part of a polygon in Γ_{i-6} and Γ_i respectively.*

(D3) For $i = 8, 14, ..., n-6$, let α_{i-1} be the polygon on Γ_{i-6} containing the triangle $\triangle p_1p_{i-1}p_i$ and let α_{i+1} be the polygon on Γ_i containing the triangle $\triangle p_1p_ip_{i+1}$, observe that α_{i-1} and α_{i+1} share the edge p_1p_i. Let $\gamma_i = \alpha_{i-1} \overline{\cup} \alpha_{i+1}$. By Observation 33, α_{i-1} contains the edge $p_{i-1}p_i$ and α_{i+1} contains the edge p_ip_{i+1}, so γ_i is well defined, and always convex.

Claim 34

$$|\mathcal{R}| = \frac{n}{2} - h(P) - 2|C_3| - |C_4| + 1.$$

Proof of Claim 34. Consider a triangulation of P by triangulating every Q_i and considering all the triangles in $\mathcal{R}$. Observe that every triangulation of Q_i has 9, 10 and 11 elements if Q_i belongs to C_5, C_4 and C_3 respectively. Since $m = |C_3| + |C_4| + |C_5|$, then

$$2n - h(P) - 2 = 9|C_5| + 10|C_4| + 11|C_3| + |\mathcal{R}| = 9m + |C_4| + 2|C_3| + |\mathcal{R}|.$$

The last equation and the fact that $m = \frac{n-2}{6}$ imply that $|\mathcal{R}| = \frac{n}{2} - h(P) + 1 - |C_4| - 2|C_3|$. ∎

Now we calculate $|\Gamma|$. Observe that if Q_i is in C_5, C_4 or C_3, then the corresponding convex decomposition will have 6, 7 or 8 elements respectively. So, when we obtain every Γ_i according to (D1) and join $m-1$ of them as described in (D3) we get $6|C_5| + 7|C_4| + 8|C_3| - (m-1) = 5m + 1 + |C_4| + 2|C_3|$ polygons. Now, considering those in $\mathcal{R}$ we have that

$$|\Gamma| = 5m + 1 + 2|C_3| + |C_4| + |\mathcal{R}| = \frac{4n}{3} - h(P) + 1.$$

∎

Corollary 1 *Let m be a positive integer, $r \in \{1,2,3,4,5\}$, and $n = 6m + 2 + r$. If P is a $\pm$ set and p_3 has label $-$, then P has a convex decomposition Γ with at most $\frac{4n}{3} - h(P) + 5$ elements.*

Proof: To obtain a convex decomposition Γ' of P we make $P' = P \backslash \{p_n, .., p_{n-r+1}\}$. Let h' be the number of vertices in $\mathrm{conv}(P')$. Then we apply Lemma 32 on P', and triangulate the interior of the region $\mathrm{conv}(P) - \mathrm{conv}(P')$, with $h' - h(P) + 2r$ triangles.

That produces a convex decomposition Γ' of P with $(\frac{4(n-r)}{3} - h' + 1) + (h' - h(P) + 2r) = \frac{4}{3}n + \frac{2}{3}r - h(P) + 1$ elements. Since $r \leq 5$, then

$$|\Gamma'| < \frac{4}{3}n - h(P) + 5.$$

∎

Corollary 2 *Let m be a positive integer, $r \in \{1,2,3,4,5\}$ and $n = 6m + 2 + r$. If P is a $\pm$ set and p_3 has label $+$, then P has a convex decomposition Γ with at most $\frac{4n}{3} - h(P) + 6$ elements.*

Proof: We will obtain a convex decomposition Γ'' of P, by making $P'' = P - \{p_2\}$, and h'' the number of vertices in $\mathrm{conv}(P'')$, applying Corollary 1 to P'', and triangulate the interior of the region $\mathrm{conv}(P) - \mathrm{conv}(P'')$, with $h' - h(P) + 2$ triangles.

With this we obtain a convex decomposition Γ'' of P, with at most $(\frac{4}{3}(n-1) - h'' + 5) + (h'' - h(P) + 2) = \frac{4n}{3} - h(P) + 6$ elements. ∎

We have the following fact.

Proposition 35 *Let γ be a convex polygon and p be a point in the interior of γ. If P is the vertex set of γ, then $P \cup \{p\}$ has a convex decomposition with 3 elements.*

Proof: Let Γ be the convex decomposition of $P \cup \{p\}$. We have that there must be an edge $e = vp$ for some $v \in P$. Otherwise the graph induced by Γ will be disconnected. Let ℓ be the line containing v and p, directed from v to p. Let $\alpha \neq \beta$ be the polygons sharing e. Since they are convex, we can assume that α is the polygon which interior lies entirely on the left half plane bounded by ℓ. Let pq be the adjacent edge to vp in $\mathrm{conv}(\alpha)$. Observe that the angle $\angle vpq > \pi$, so pq is not an edge of β. We must have another polygon sharing pq with α, so $|\Gamma| \geq 3$.

To get a convex decomposition with three elements, we pick an element $u \in P$, and get the triangulation whith all its elements sharing u, see Figure 4.10 (left). Observe that p will be an interior point of a triangle $\triangle uvv'$, where v and v' are adjacent in $\mathrm{conv}(P)$. Let ℓ' be the line containing u and p, directed from u to p. Let L be the set of elements in P on the open left half plane bounded by ℓ'. Similarly, we make R, the set of elements in P, on the open right half plane bounded by ℓ'. Observe that $\{\mathrm{conv}(R \cup \{u,p\}), \mathrm{conv}(L \cup \{u,p\}), \triangle pvv'\}$ is a convex decomposition of $P \cup p$. See Figure 4.10 (right).

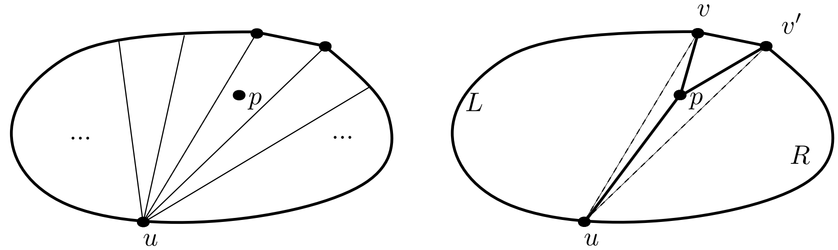

Figure 4.10: Getting a convex decomposition of $P \cup \{p\}$ with three elements.

∎

We proceed now to prove our main theorem.

Theorem 36 *P has a convex decomposition with at most $\frac{10}{7}n - h$ elements, where h is the number of vertices in* $\mathrm{conv}(P)$.

Proof: We obtain Graham's triangulation $\Gamma_G(P)$, and then modify it as indicated in (C1) and (C2). Let h be the number of vertices in $\mathrm{conv}(P)$ and Γ be the resulting convex decomposition. By Lemma 31, $|\Gamma| = n + k - h$. Then, if $k \leq \frac{3n}{7}$, we are done. Also, if $k = \frac{n}{2}$, then P is a $\pm$ set. So, by Lemma 32, it has a convex decomposition with $\frac{4n}{3} - h + 6$ elements, and again we are done.

In case that $\frac{3n}{7} < k < \frac{n}{2}$, we obtain a $\pm$ set P' from P in following way:

Let C denote the set of vertices of $\mathrm{conv}(P)$. Let s and t be the greatest and smallest index of the elements in $C \cap A'_1$ and $C \cap (A'_k \setminus \{p_1\})$ respectively. See Figure 4.11 (center).

For $i = 2, 3, \ldots, k-1$ and $j = 1, 2, \ldots, k-1$, we take an element $q_i \in A_i$ and one $r_j \in B_j$ in such a way that $P' = \{p_1, p_s, r_1, q_2, r_2, \ldots, q_{k-1}, r_{k-1}, p_t\}$ is a $\pm$ set. See Figure 4.11 (right). Let h' be the number of vertices in $\mathrm{conv}(P')$. Observe that $|P'| = 2k$ and $h' \leq h - (s-2) - (n-t)$.

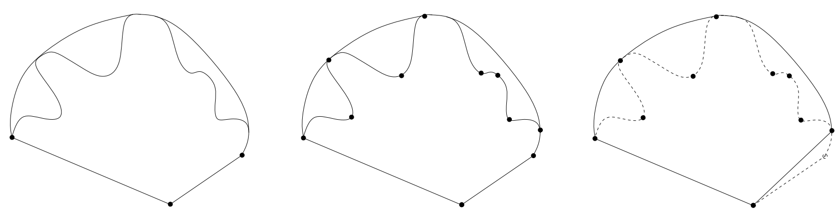

Figure 4.11: P and its $\pm$ collection associated. In this case $s > 2$ and $t = n$.

Let Γ' be the convex decomposition of P' produced by Lemma 32, and $S = P \backslash P'$. To get a convex decomposition Λ of P we start from Γ' and add every element in S one by one, also adding on Γ' the polygons that Proposition 35 yields. Every element in S, when is added, increases in 2 the number of polygons to the current decomposition, so $|\Lambda| \leq \frac{4}{3}2k - h' + 6 + 2|S|$. As $|S| = n - 2k$, then $|\Lambda| \leq 2n - \frac{4}{3}k - h' + 6$. Since $k \geq \frac{3n}{7}$, we obtain that

$$|\Lambda| \leq \frac{10n}{7} - h' + 6.$$

If $s > 2$ (or $t < n$) we make α the polygon $p_1 p_2 \ldots p_s$ ($\beta = p_1 p_t \ldots p_n$ respectively), and add them to Λ. We will have that

$$|\Lambda| \leq \frac{10n}{7} - h + 6.$$

∎

Chapter 5

Future work

5.1 Chromatic number of $D(P)$

1. It is known that a maximal thrackle, with vertices in P, has at most n edges, those we find in sets in convex position. That is, we find the largest thrackles when the point set is in convex position. We strongly believe that the lower bound is given by C_n. That is

$$L(n) = f(n).$$

2. Let $P = V_{5,5} \cup V_{3,3}$. Such that, the elements in $V_{5,5}$ are placed in a way that $\text{conv}(V_{5,5})$ is containing $V_{3,3}$ in its interior, in a connected region of $\mathbb{R}^2 \setminus D(V_{5,5})$. The elements in $V_{3,3}$ are placed in such a way that no line containing two of its vertices intersects $\text{conv}(A_5) \cup \text{conv}(B_5)$. See Figure 5.1. This point set configuration is interesting because it proves that $n_k \geq 17$, since it has no convex hexagons. In Figure 5.1 we show the point set $\{(-54, 92), (-48, 89), (-40, 87), (48, 89), (54, 92), (-12, -1.5), (-9, -1.25), (-6, -1.5), (6, 1.5), (9, 1.25), (12, 1.5), (-81, -80), (-74, -76), (-72, -75), (74, -76), (81, -80)\}$.

Applying the main idea in Chapter 5, we can prove that $\chi(D(P)) \leq f(5)+11$. On the other hand, if we color optimally both of $V_{5,5}$ and $V_{3,3}$, we obtain that $\chi(D(P)) \geq f(3) + 3 + f(5) + 5$. Hence

$$12 \leq \chi(D(P)) \leq 14.$$

We are interested in calculate the exact value of $\chi(D(P))$.

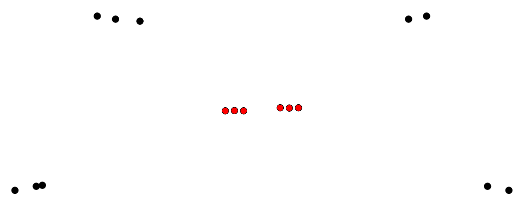

Figure 5.1: $P = V_{5,5} \cup V_{3,3}$.

5.2 Convex decompositions

1. We want to improve the current bound, or at least give another specific convex decomposition that reaches the Hosono's bound.

2. Assume that n is odd. Observe that if P is in convex position, then we can find a convex decomposition with $\frac{n-3}{2}$ quadrilaterals and a triangle. We want to show that, in general, these number of quadrilaterals remains. In other words, we want to prove that, in any point set P, we can find a family Q, of convex quadrilaterals, satisfying $\mathcal{CD}1$, $\mathcal{CD}2$ and $\mathcal{CD}3$ such that $|Q| \geq \frac{n-3}{2}$.

In terms of convex decompositions, we want to prove that always we can find a convex decomposition Γ consisting only in a set T, of triangles, and a set Q, of quadrilaterals, such that $|Q| \geq \frac{n-3}{2}$.

With the given convex decomposition in Chapter 6, we can prove that, if m_a and m_b are the number of A_i and B_i sets, with odd and even cardinality, respectively, then

$$|Q| = \frac{n-3}{2} + \frac{m_a - m_b}{2}.$$

Bibliography

[1] O. Aichholzer and H. Krasser. *The point set order type data base: A collection of applications and results.* In Proc. 13th Annual Canadian Conference on Computational Geometry CCCG 2001, pages 17-20, Waterloo, Ontario, Canada, 2001.

[2] M. Aigner, G. Ziegler *Proofs from THE BOOK*, 4th ed. Springer 2010.

[3] G. Araujo, A. Dumitrescu, F. Hurtado, M. Noy, and J. Urrutia. *On the chromatic number of some geometric type Kneser graphs.* Computational Geometry, 32(1):59-69 (2005).

[4] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications.* North–Holland, New York (1982).

[5] R. Diestel, *Graph Theory.* 2nd. ed. Springer–Verlag (2000)

[6] P. Erdös, G. Szekeres, *A combinatorial problem in geometry.* Compositio Math., 2:463–470, 1935.

[7] P. Erdös, *On Sets of Distances of n Points*. The American Mathematical Monthly, 53(5):248-250 (1946).

[8] R. Fabila-Monroy, C. Hidalgo-Toscano, J. Leaños, M. Lomelí-Haro *The Chromatic Number of the Disjointness Graph of the Double Chain.* Discrete Mathematics & Theoretical Computer Science, vol. 22 no. 1, (2020) DOI: 10.23638/DMTCS-22-1-11

[9] R. Fabila-Monroy, and D. R. Wood (2012). *The chromatic number of the convex segment disjointness graph.* In Computational Geometry. Springer Berlin Heidelberg.

[10] J. Galtier, F. Hurtado, M. Noy, S. Perennes, J.Urrutia, *Simultaneous Edge Flipping in Triangulations.* Int. J. Comp. Geometry, 13, no. 2, 113-133, (2004).

[11] J. García-López, C. Nicolás, *Planar Point Sets with Large Minimum Convex Partitions*. 22nd European Workshop on Computational Geometry, Delphi, Greece, pp. 51-54, 2006.

[12] T. Gerken *Empty convex hexagons in planar point sets.* (English summary) Discrete Comput. Geom. 39 (2008), no. 1-3, 239-272.

[13] H. Harborth *Konvexe Fünfecke in ebenen Punktmengen.* Elem. Math., 333:116–118, 1979

[14] J.D. Horton, *Sets with no empty convex 7–gons*, Canada Math. Bull. 26 (1983) 482–484.

[15] K. Hosono. *On convex decomposition of a planar point set.* Discrete Mathematics, 309(6):1714-1717, 2009.

[16] F. Hurtado, M. Noy, J.Urrutia, *Flipping edges in triangulations.* Discrete and Computational Geometry, 22,.333-346 (1999). Also in Proc. 12 ACM Symposium on Computational Geometry, May 24-26 1996, 214-223.

[17] J. Jonsson (2011). *The exact chromatic number of the convex segment disjointness graph.* DOI 10.1007/978-3-642-34191-5_7.

[18] V. Neumann, E. Rivera–Campo, J. Urrutia, *A Note on Convex decompositions.* Graphs and Combinatorics Vol. 20, no. 2, pp. 223-231, (2004).

[19] M. Overmars, *Finding sets of points without empty convex 6-gons.* Discrete Comput. Geom. 29, 153?158 (2003)

[20] G. Tóth, P Valtr, *Note on the Erdös-Szekeres theorem.* Discrete Comput Geom (1998) 19: 457. https://doi.org/10.1007/PL00009363

[21] J. Urrutia *Open problem session.* 10^{nth} Canadian Conference on Computational Geometry, McGill University, Montreal, 1998.