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A Nullity Axiom for Games in Partition Function Form

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¡Viva la libertad carajo!

Javier Milei

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Abstract

This work focuses on game theory, with an emphasis on the unique characterization of Myerson's value in partition function games. Drawing inspiration from Shapley's foundations, Myerson proposed a unique solution characterized by three axioms: linearity, symmetry, and the carrier axiom. The research aimed to determine if an equivalent characterization was possible using four axioms: linearity, symmetry, efficiency, and nullity. As far as we know, there is not such characterization in the current literature. A proposal for a null player is introduced, revealing that the derived characterization is not unique. Instead, a family of parameterized solutions is presented, demonstrating the diversity of potential outcomes in this context.

Keywords

1. Game theory
2. Myerson's value
3. Games in partition function form
4. Axioms (linearity, symmetry, efficiency, nullity, carrier)
5. Null player
6. Coalitional Games
7. Shapley's value

Resumen

Este trabajo se centra en la teoría de juegos, con énfasis en la caracterización única del valor de Myerson en juegos de función de partición. Inspirado en los fundamentos de Shapley, Myerson propuso una solución única caracterizada por tres axiomas: linealidad, simetría y el axioma del portador. La investigación tuvo como objetivo determinar si era posible una caracterización equivalente utilizando cuatro axiomas: linealidad, simetría, eficiencia y nulidad. Hasta donde sabemos, no existe tal caracterización en la literatura actual. Se introduce una propuesta para un jugador nulo, revelando que la caracterización derivada no es única. En cambio, se presenta una familia de soluciones parametrizadas, que demuestran la diversidad de resultados potenciales en este contexto.

Palabras clave:

1. Teoría de juegos
2. Valor de Myerson
3. Juegos en forma de función de partición
4. Axiomas (linealidad, simetría, eficiencia, nulidad, portador)
5. Jugador nulo
6. Juegos coalicionales
7. Valor de Shapley

Contents

List of Symbols

$\wedge$	coarsest common refinement, page 7
$\vert\lambda\vert$	number of elements of λ, page 8
$\vert\langle partition\rangle\vert$	partition size, page 6
$\vert\langle set\rangle\vert$	set cardinality, page 5
$[\langle partition\rangle]$	partition into singletons, page 6
$\emptyset$	empty set, page 5
λ,γ	a partition nonnegative integer of n, page 8
$\lambda_{\mathscr{P}}$	integer partition associated with $\mathscr{P}$, page 8
λ_i	positive integer element of λ, page 8
λ°	the set of the numbers determined by the $\lambda_i's$, page 8
π	a permutation, page 8
πf	the permutation of a function, page 8
$\pi^{i,j}$	the permutation that exchanges players i and j, page 12
$\pi(S,\mathscr{P})$	the permutation of an embedded coalition, page 8
υ	characteristic function, page 9
υ_T	the unanimity game over T, page 12
Γ	the set of characteristic function form games, page 9
$\Pi^i(\mathscr{P})$	the set of partitions that may arise after i leaves its current coalition, page 7
$\tilde{\Pi}(N)$	set of partitions of N, page 6
$\Lambda(n)$	the set of all partitions of n, page 8
S_n	the set of permutations of a set, page 8
$i,j,\ldots$	set element, page 5
m_δ^λ	multiplicity of the element $\delta\in\lambda$ in partition λ, page 8
w	a partition function, page 16
$\overline{w}$	the efficient cover of a partition function, page 36
B_n	n-th Bell-number, page 6
$C,S,T,\ldots$	coalition, page 5

F_n	the set (of cardinalaty) that counts the number of derived equations from nullity restrictions in the expresion of the LSE solutions, page 28
$\tilde{\Gamma}$	the set of partition function form games, page 16
$\overline{S}$	complement of S, page 5
$S_i, S_j, \ldots$	elements of a partition of S, page 6
S^{+i}	set S with the element i, page 5
S^{-i}	set S without the element i, page 5
$[S]$	The partitioning of the set S into singletons, page 6
	$\mathscr{E}$
set of embedded coalitions, page 6	
$\mathscr{P}, \mathscr{Q}, \mathscr{R}$	a partition, page 6
$\mathscr{P}(i)$	the element of partition P containing i, page 6
$\mathscr{P}^i(T)$	the partition $\mathscr{P}$ after i left his coalition and joined T, page 7
$\mathscr{P}(S)$	a partition of set S, page 5

1 Introduction

"Los fallos de mercado no existen. O bien se trata de una anomalía que se ha generado por la intervención siempre violenta del Estado (fallo del gobierno), o, se trata de un fallo del analista." [Market failures do not exist. Either it is an anomaly generated by the always violent intervention of the State (government failure), or it is a failure of the analyst.]

Javier Milei

Game theory, a powerful mathematical tool, finds fundamental applications in various industries by analyzing situations of conflict and cooperation. Its purpose is to provide a deep understanding and normative guidelines for the rational behavior of economic agents facing strategic decisions and complex social interactions.

In industry, this theory is used to model and solve economic and political challenges, offering valuable insights into markets, competition, and strategic alliances. Additionally, in operations research, it is employed to optimize processes and enhance decision-making.

In game theory, we encounter two fundamental approaches: cooperative games and non-cooperative games. It is essential to grasp this distinction, which arises from the motivations and dynamics of participants in a game.

In non-cooperative games (which we will not address), each player makes decisions independently, without forming agreements with other players. In this context, the primary objective of each player is to maximize their individual benefit, regardless of the actions or outcomes of other players.

Conversely, in cooperative games, participants actively seek to negotiate and collaborate, either as a complete group or in subsets, with the aim of achieving greater joint benefits. Thus, the most prominent theoretical difference lies in the concept of *strategy* in non-cooperative games and the concept of *coalition* in cooperative games.

In this work, we will explore two types of cooperative games: characteristic function form games and partition function form games. In these models, we will assume that players have the ability to reach agreements and mutually compensate each other through the transfer of utility, using, for example, a perfectly divisible resource. If such a resource is not perfectly divisible, it is considered that players have access to another compensatory resource that is divisible.

Next, an example is presented that illustrates the issue of profit distribution in cooperation situations among different agents. This example lays the foundation for the formalization of the theory and the axiomatic analysis of various proposed solutions in the literature, all within the context of the current work.

Example 1.1 (Market competition) *Let's consider the following game, which describes a scenario where we have* 3 *companies competing for a market. When players* 1, 2, *and* 3 *act individually, each of them obtains a value of* 50 *monetary units. However, if any of them join forces, they can capture a larger share of the market and thus gain more profit, affecting the other player. Finally, if the grand coalition is formed, they capture the entire market and achieve the maximum profit of* 200 *monetary units.*

Partition	**Value**		
{1},{2},{3}	50	50	50
{1,2},{3}	120	40	
{1,3},{2}	125	45	
{2,3},{1}	130	10	
{1,2,3}	200		

This example, which serves as a compelling introduction to the work we are about to delve into, sheds light on a common challenge in cooperative game theory: the struggle to ensure that every player has an incentive to collaborate.

As we progress in this work, we will explore various scenarios, with a particular focus on those where one or more players may not be productive. In such cases, the need arises to find an appropriate method for fairly distributing gains or costs. This implies that contributing players would expect their compensation to be the same, regardless of whether the "non-productive player" is part of their group or not. This aspect adds a layer of complexity and relevance to cooperative game theory as it addresses the quest for fair and efficient solutions across a variety of industrial and social contexts.

So, with this example as our starting point, we will delve into both types of cooperative games.

It has been shown that Myerson's value can be characterized by the axioms of linearity, symmetry, and carrier (similarly to Shapley's value), but it was not clear if a unique characterization could be found using the axioms of linearity, symmetry, nullity, and efficiency (as is the case with Shapley). This characterization, based on a "null player," is crucial for understanding how benefits are distributed in cooperation situations where players can form coalitions.

1.1 State of the Art

Other authors have proposed different definitions to characterize a player as null in partition function form games. These definitions vary in their conditions and approaches. For example, Pham Do and Norde [8] defined a player as null if their contribution to any coalition is zero, and if by changing coalitions, the total wealth is not affected. Macho-Stadler [11] proposed a similar definition, but without the condition that the null player does not generate value by themselves. Hafalir [12] introduced the concept of an efficient-covering null player and compares how what he calls an "efficient partition" behaves instead of the original partition, where he explicitly assumes that the remaining players in the coalition are individuals (individual players without coalitions). Bolger [13] defined a player as null in the strong sense if their transfer between coalitions does not affect the wealth of any coalition. Skibski [16] introduced the concept of a null player with constant marginality, which characterizes such a player as one whose contribution to wealth is consistent and balanced in all possible partitions of the player set.

1.2 Our Contribution

In our work, we have proposed a definition of a null player that focuses on the redistribution of wealth in the game when the null player is present or absent in a coalition. Our definition evaluates how the partition $\mathscr{P}$ changes when the null player is involved in a coalition and when they are not. Unlike other definitions, our proposal does not establish specific conditions on the value of the null player or their impact on other coalitions. We have analyzed how our definition compares with other proposals in the literature and highlighted the differences in approach and requirements to be considered a null player.

1.3 Thesis Structure

In the next chapter, we will focus on establishing the necessary language and mathematical notation in sets, which will be fundamental for the further development of our work. This chapter lays the foundation for understanding the concepts and methods we will use in the following chapters.

In the third chapter, we will review the existing literature on cooperative games. In this sense, the content of this chapter will not be original, as our goal is to analyze and compile the fundamental concepts and axioms that underlie both characteristic function form games and partition function form games. This review will allow us to understand the theoretical context in which our work is inserted and will serve as a reference to contrast our results with those of other researchers.

In the four and final chapter, we will delve into the central problem of the thesis. We will begin by proposing a definition of the nullity axiom based on Myerson's carrier axiom. Then, we will explore the possibility of finding a unique solution that satisfies the axioms of Linearity, Symmetry, Efficiency, and Nullity (LSEN). We will discover that our notion of a null player generates an infinite variety of solutions, and that Myerson's value represents a specific case within this set. We will characterize this set of solutions that satisfy LSEN, which will allow us to obtain a deep understanding of its structure and properties.

Finally, we will summarize the key findings and discuss the implications of our work in the context of non-cooperative game theory.

2 Terminology and Notation

"Los hombres libres actúan a través del mercado, los represores a través del Estado." [The free men act through the market, the oppressors through the State.]

JUAN RAMÓN RALLO

Game theory relies on fundamental mathematical concepts. We follow established literature and conventions but need to establish a solid foundation in mathematical language and notation (Kóczy [1]). In the following sections, we introduce key mathematical building blocks and their notation.

2.1 Sets

Sets collect distinct objects: numbers, players, etc. They are denoted by capital letters $R, S, T, \dots$ and the elements of sets by lower case $i, j, \dots$. $N = \{1,\dots,n\}$ denotes a finite set of positive natural numbers (or players starting from the next chapter). We write $i \in S$ to denote that i is an element of S. Let $S \subseteq T$ denote that S is a subset of T and $S \subset T$ if $S \subseteq T$ but $S \neq T$. Also, let = $2^N = \{T \mid T \subseteq N, T \neq \emptyset\}$ denote the power set, the set of subsets of N.

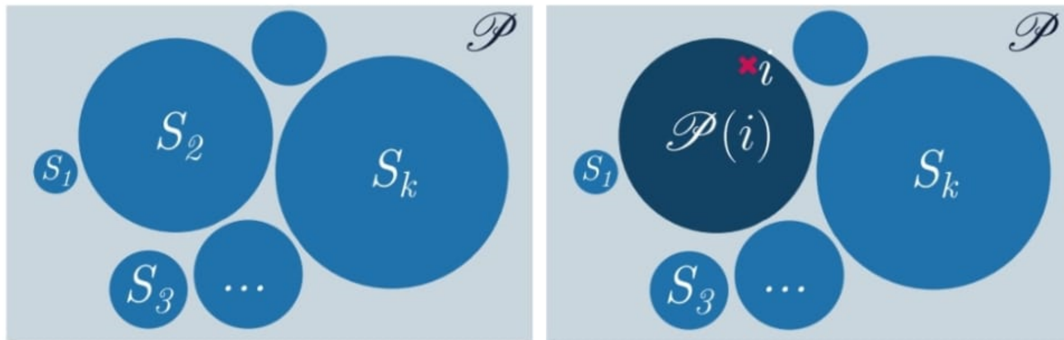
The difference of sets S and T is $S \setminus T = \{i \mid i \in S, i \notin T\}$. Let $S^{-i} = S \setminus \{i\}$ and $S^{+i} = S \cup \{i\}$. The complement of a set is denoted by a bar, $\overline{S} = N \setminus S$, where N is the universe of objects. The cardinality or size of a set S, denoted by $|S|$, is the number of elements in S. A set consisting of a single element is called a singleton; the set containing no elements is the empty set, denoted by $\emptyset$.

2.1.1 Partitions

A partition of S, is a set $\{S_1, S_2, \dots, S_m\}$ of subsets of S such that

$$\bigcup_{i=1}^{m} S_i = S,\ S_j \neq \emptyset\, \forall j,\ S_j \cap S_k = \emptyset\, \forall j \neq k$$

Figure 2.1a shows such a partition: the box represents the coalition S, and S_i represents group members of S.



(a) The partition of S into disjoint $S_1, \ldots, S_k$. Partitions cover the entire S, but since S has finite elements, they are not everywhere.

(b) The coalition embedded in $\mathscr{P}$ containing player i is denoted by $\mathscr{P}(i)$.

Figure 2.1 A partition $\mathscr{P}$.

The set of partitions of N is denoted by $\tilde{\Pi}(N)$. The number of partitions of a set with $n \in \mathbb{N}$ elements is given by the Bell-number B_n (Bell [2]). Clearly, $B_0 = 1$ and $B_1 = 1$; for larger values, the number can be calculated recursively using the following formula:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{2.1}$$

Bell-numbers were originally called *exponential numbers*.

Table 2.1 The first Bell-numbers expressing the number of partitions.

n	0	1	2	3	4	5	6	7	8	9
B_n	1	1	2	5	15	52	203	877	4140	21147

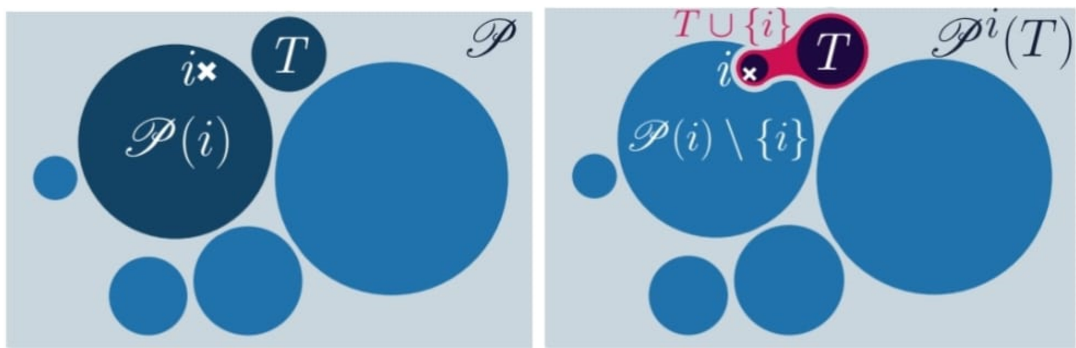
The size of a partition $\mathscr{P}$, denoted by $|\mathscr{P}|$, is the number of elements of $\mathscr{P}$. The partitioning of the set S into singletons is denoted by $[S] = \{\{j\} | j \in S\}$.

We say that a coalition C is embedded in partition $\mathscr{P}$ if $C \in \mathscr{P}$; by embedded coalition we mean a pair $(S, \mathscr{P})$, the coalition together with the embedding partition. The set of embedded coalitions is

$$\mathscr{E} = \{(C, \mathscr{P}) \,|\, C \in \mathscr{P}, \mathscr{P} \in \tilde{\Pi}(N)\}$$

The coalitions embedded in $\mathscr{P}$ are the elements, or blocks or atoms, of the partition $\mathscr{P}$.

For a set S, an element $i \in S$, and a partition $\mathscr{P} = \{S_1, S_2, \ldots, S_k\}$ of S, let $\mathscr{P}(i) \in \mathscr{P}$ such that $i \in \mathscr{P}(i)$ (see Figure 2.1b).



(a) Partition $\mathscr{P}$ containing $T \not\ni i$. **(b)** Partition $\mathscr{P}^i(T)$ after i joins T.

Figure 2.2 $\mathscr{P}^i(T)$ emerges when i leaves $\mathscr{P}(i)$ to join coalition T.

Also, let $\mathscr{P}^i(T) = \mathscr{P} \setminus \{\mathscr{P}(i), T\} \cup \{\mathscr{P}(i)^{-i}, T^{+i}\}$ denote the partition that results when player i leaves its current coalition and joins coalition T (see Figure 2.2). Let $\Pi^i(\mathscr{P}) = \{\mathscr{P}^i(T) | T \in \mathscr{P} \cup \{\emptyset\}, T \neq \mathscr{P}(i)\}$ denote the set of partitions that may result when player i leaves his current coalition and either joins another coalition or stays single.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two partitions of the same set. The partition $\mathscr{R}$ is a common refinement of $\mathscr{P}$ and $\mathscr{Q}$ if $\mathscr{R}$ is a refinement of both $\mathscr{P}$ and $\mathscr{Q}$. The coarsest common refinement is denoted by $\mathscr{P} \wedge \mathscr{Q}$. The operation $\wedge$ as follows (Myerson [3]):

$$\mathscr{P} \wedge \mathscr{Q} = \{S \cap T \mid S \in \mathscr{P}, T \in \mathscr{Q}, S \cap T \neq \emptyset\} \tag{2.2}$$

$\mathscr{P} \wedge \mathscr{Q}$ is an intermediate partition that captures the intersection of the two original partitions $\mathscr{P}$ and $\mathscr{Q}$.

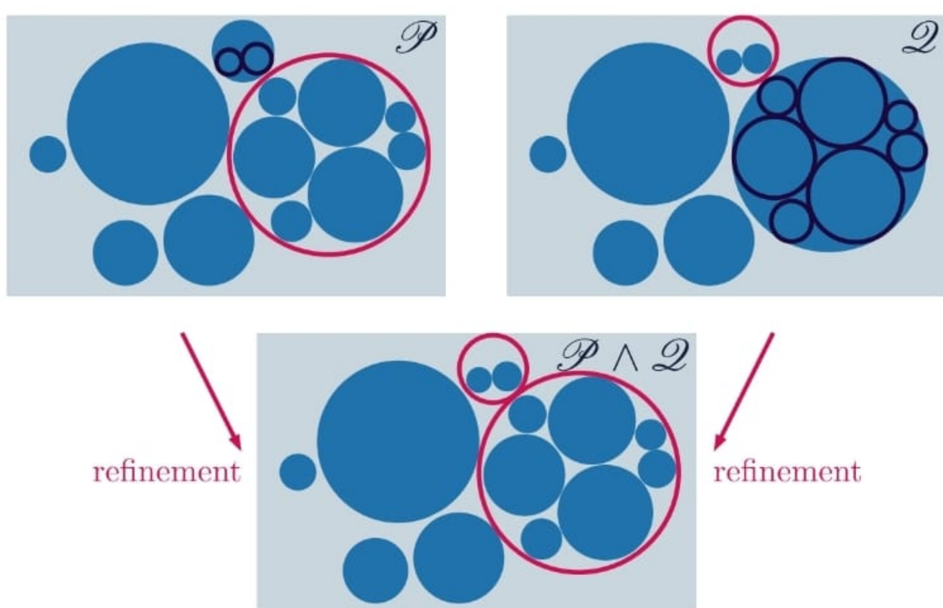


Figure 2.3 Partitions $\mathscr{P}$ and $\mathscr{Q}$ are mutually incomparable: neither is a refinement of other. Operation $\mathscr{P} \wedge \mathscr{Q}$ (2.2).

2.1.2 Permutations

A permutation is a bijection of a set onto itself. For a set N, the set of permutations is $S_n = \{\pi : N \to N \,|\, \pi \text{ is bijective}\}$, where a generic element is denoted by π. The permuted image of a particular $i \in S$ element of S is πi. It is natural to consider permutations of sets, partitions, or even functions: given a permutation $\pi \in S_n$, the permutation of the set $S \subseteq N$ is $\pi S = \{\pi i \,|\, i \in S\}$. The permutation of a partition $\mathscr{P} \in \Pi(S)$ follows naturally: $\pi\mathscr{P} = \{\pi S_i \,|S_i \in \mathscr{P}\}$. The permutation of an embedded coalition is $\pi(S_i,\mathscr{P}) = (\pi S_i, \pi\mathscr{P})$. For $\pi \in S_n$ and $x = (x_1,...,x_n) \in \mathbb{R}^n$, $\pi x = (x_{\pi(1)},...,x_{\pi(n)})$. The action of S_n on $\mathbb{R}^n$ with $x = (x_1,...,x_n) \in \mathbb{R}^n$ is given by $\pi x = (x\pi(1),...,x\pi(n))$. Given some set Ξ with a generic element $X \in \Xi$, the permutation of a function $f : \Xi \to \mathbb{R}$ satisfies

$$(\pi f)(X) = f(\pi X) \tag{2.3}$$

2.2 Integer Partitions

A partition nonnegative integer is way of expressing it as the unordered sum of the other positive integers, and it is often writen in tuple notation. Formally, $\lambda = [\lambda_1,\lambda_2,...,\lambda_l]$ is a partition of n iff $\lambda_1,\lambda_2,...,\lambda_l$ are positive integers and $\lambda_1 + \lambda_2 + ... + \lambda_l = n$. Two partitions which only differ in the order of their elements are considered to be the same partition. The set of all partitions of n will be denoted by $\Lambda(n)$, and, $\lambda \in \Lambda(n)$. $|\lambda|$ is the number of elements of λ.

For example, the partitions of n = 4 are [1,1,1,1], [2,1,1], [2,2], [3,1] and [4]. Sometimes we will abreviate this notation by dropping the commas, so [2,1,1] becomes [211].

If $\mathscr{P} \in \Pi(N)$, there is a unique partition $\lambda_{\mathscr{P}} \in \Lambda(n)$, associated with $\mathscr{P}$, where the elements of $\lambda_{\mathscr{P}}$ are exacly the cardinalities of the elements of $\mathscr{P}$. In other words, if $\mathscr{P}$ = $\{S_1,S_2,...,S_m\} \in \Pi(N)$, then $\lambda_{\mathscr{P}}$ = $[|S_1|,|S_2|,...,|S_m|]$.

For a given $\lambda \in \Lambda(n)$, we represented by λ° the set of the numbers determined by the $\lambda_i' s$ and for $\delta \in \lambda^\circ$, we denoted by m_δ^λ the multiplicity of δ in partition λ with the condition of m_0^λ = 1. So, if λ = [4,2,2,1,1,1], then $\lambda^\circ = \{1,2,4\}$ and m_1^λ = 3, m_2^λ = 2, m_4^λ = 1.

Let $\lambda, \gamma \in \Lambda(n)$ be partitions such that $\gamma^\circ \subseteq \lambda^\circ$, we define the difference $\lambda - \gamma$ as a new partition obtained from λ by removing the elements of γ. For example, [4,3,2,1,1,1] - [3,1,1] = [4,2,1].

3 Cooperative Games

"La «justicia social» sólo tiene sentido en un fantasmagórico mundo estático en el que los bienes y servicios se encuentren dados y el único problema que pueda plantearse sea el de cómo distribuirlos." [«Social justice» only makes sense in a ghostly static world where goods and services are already given, and the only problem that can arise is how to distribute them.]

ISRAEL M. KIRZNER

In this chapter, we will delve into two categories of cooperative games: those following a characteristic function approach and those based on partition function.

3.1 Characteristic Function Form Games

Here, we will present the fundamental elements for defining the classical model of transferable utility games, hereinafter referred to as ***TU games***.

Definition 3.1 (*TU* Game) *A game in characteristic function form is a pair (N, υ), consisiting of a set N of players and a characteristic function*

$$\upsilon : 2^N \to \mathbb{R}$$

that assigns to each non-empty subset S of N a real number $\upsilon(S)$ representing the joint gains of the players. The function satisfies the condition $\upsilon(\emptyset) = 0$, which means that the empty set does not generate any gains.

The function $\upsilon : 2^N \to \mathbb{R}$ is referred to as a traditional game in this context, as the set of players remains constant throughout this work. We define the space of traditional games as

$$\Gamma = \{\upsilon : 2^N \to \mathbb{R} \mid \upsilon(\emptyset) = 0\}$$

It is essential to note that this set of traditional games, with the player set N, forms a vector space over the field of real numbers. Given $\upsilon_1, \upsilon_2 \in \Gamma$ and $\alpha \in \mathbb{R}$, we can define the sum $\upsilon_1+\upsilon_2$ and the product $\alpha\upsilon_1$, in Γ, in the usual form, i.e,

$$(\upsilon_1+\upsilon_2)(S)=\upsilon_1(S)+\upsilon_2(S) \quad \textit{and} \quad (\alpha\upsilon_1)(S)=\alpha\upsilon_1(S)$$

These operations are fundamental to understanding and modeling cooperative situations among players, allowing the creation of new games from existing ones.

The vector sum of cooperative games combines two existing games into a new one. In this process, for each coalition of players, the joint payoff in the new game is the sum of the joint payoffs of the original games in that coalition.

On the other hand, the scalar product of a cooperative game υ by a real number λ results in a new cooperative game in which the payoffs for each coalition S are scaled by the value of λ. In other words, if you have a cooperative game υ and a real number λ, the resulting game $\lambda\upsilon$ assigns to each coalition S the payoff it would receive in υ multiplied by λ.

When a coalition $S\subseteq N$ is formed, where players collaborate together, the question of how to fairly divide the payment $\upsilon(S)$ obtained among its members arises. To address this dilemma, we introduce the concept of payment distribution, which assigns to each pair (N,υ) a vector x_N in $\mathbb{R}^n$. Each component of this vector, x_i, represents the payment allocated to player i within the set N.

This perspective leads us to define an operator φ, which operates on Γ, assigning values to each of them,

$$\varphi:\Gamma\to\mathbb{R}^n$$

$(i,\upsilon)\mapsto\varphi_i(\upsilon)$, is therefore a function that, for any player set N, associates each player in N with a real number. However, determining how to measure each player's contribution to the attainment of the value $\upsilon(S)$ presents a fundamental challenge. The goal is to find a function φ that satisfies *rational* axioms and properties for allocating fair payments to the players.

Ultimately, it is demonstrated that there exists a unique operator that satisfies these axioms. In summary, the central problem addressed in these types of games revolves around finding a solution that rationally distributes payments among the players.

3.1.1 Players Who do not Contribute to the Game

Before introducing the axioms that form the basis of a unique solution concept, we need additional definitions. The treatment of players who do not contribute to the game often raises interesting questions, so we begin by identifying them. In the early works of game theory, authors referred to a universe of players, denoted as N, within which exists a finite coalition of relevant players (Narahari, 2012 [4]).

Definition 3.2 (Carrier) *A coalition T is considered a carrier of a coalitional game $\upsilon\in\Gamma$ if,*

$$\upsilon(S\cap T)=\upsilon(S) \quad \forall S\subseteq N$$

If T is a carrier coalition and $i\notin T$, then:

$$\upsilon(\{i\})=\upsilon(\{i\}\cap T)=\upsilon(\emptyset)=0$$

If T is a carrier of $\upsilon\in\Gamma$, all players $j\in\overline{T}$ are referred to as *null players* in υ, as their inclusion in any coalition does not affect the coalition's value. Additionally, for any $S\subseteq N$

and $i \notin T$:

$$\upsilon(S^{+i}) = \upsilon(S^{+i} \cap T) = \upsilon(S \cap T) = \upsilon(S)$$

In summary, T includes all influential players (although it may also include non-influential players, it does not exclude influential ones). If T is a carrier coalition and $i \in N$, then for any $S \subseteq N$:

$$\upsilon(S) = \upsilon(S \cap T) = \upsilon(S \cap T^{+i})$$

This means that T^{+i} is also a carrier coalition for any $i \in N$. In fact, the set N is always a carrier coalition, implying:

$$\upsilon(T) = \upsilon(N)$$

However, it is possible that no proper subset of N is a carrier.

A player $i \in N$ is considered null in $\upsilon \in \Gamma$ if and only if the set N^{-i} is a carrier of υ. Then, $\forall S \subseteq N$

1. If $i \in S$,

$$\upsilon(S) = \upsilon(N^{-i} \cap S) = \upsilon(S^{-i})$$

2. If $i \notin S$,

$$\upsilon(S) = \upsilon(N^{-i} \cap S) = \upsilon(S)$$

It is important to note that if $i \notin S$, then $S = S^{-i}$. This leads us to the following definition

Definition 3.3 (Null Player) *A player $i \in N$ is deemed null in $\upsilon \in \Gamma$ if for all $S \subseteq N$*

$$\upsilon(S) = \upsilon(S^{-i})$$

or equivalently, for all $S \not\ni i$,

$$\upsilon(S^{+i}) = \upsilon(S)$$

Now, if T is a carrier of $\upsilon \in \Gamma$ and $i \notin T$, then i is a null player in υ.

Proof. *Suppose that $T \subseteq N$ is a carrier in $\upsilon \in \Gamma$, then*

$$\upsilon(S) = \upsilon(S \cap T) \quad \forall S \subseteq N$$

Let $i \in N$ such that $i \notin T$ and $i \in S$:

$$\upsilon(S^{-i}) = \upsilon(S^{-i} \cap T) = \upsilon(S \cap T) = \upsilon(S)$$

so, i is null in υ. ■

In the literature, the concept of a *dummy player* is frequently encountered and is often confused with the concept of a null player. Essentially, we will highlight the differences between a null player and a *dummy player*.

Definition 3.4 (Dummy Player) *A player $i \in N$ is considered dummy in $\upsilon \in \Gamma$ if for all $S \subseteq N$*

$$\upsilon(S^{+i}) = \upsilon(S^{-i}) + \upsilon(\{i\})$$

A null player never contributes anything, and a dummy player has its own value but does not add anything beyond that. Null players are totally worthless, while dummy players may be valuable but have very poor cooperation skills. All null players are also dummy players but not vice versa. Dummy players belong to each carrier, while for null players there are carriers that do not contain them. The two terms are often confused.

The next property, symmetry deals with players who can be freely exchanged. For the definition we need some additional notation. Let $\pi^{ij} \in S_n$ denote the permutation that exchanges players i and j: $\pi(i) = j$, $\pi(j) = i$ and $\pi(k) = k$ for all $k \notin \{i, j\}$.

Definition 3.5 (Symmetric Players) *Players i and j are symmetric in $\upsilon \in \Gamma$ if $\upsilon(S) = \upsilon(\pi^{ij} S)$*

Certain characteristic function form games deserve special attention. A game $\upsilon \in \Gamma$ is simple if $\upsilon(C) \in \{0,1\}$ for all $C \subseteq N$. *Unanimity games* are a special group of simple games that are symmetric with respect to non-null players (Kóczy [1]).

Definition 3.6 (Unanimity game) *For any subset $T \subseteq N$ where $T \neq \emptyset$, the unanimity game υ_T is given by*

$$\upsilon_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise} \end{cases}$$

Example 3.1 *Let $N = \{1,2,3\}$. Let $\upsilon_{\{1\}}, \upsilon_{\{2\}}, \upsilon_{\{3\}}, \upsilon_{\{1,2\}}, \upsilon_{\{1,3\}}, \upsilon_{\{2,3\}}, \upsilon_{\{1,2,3\}}$ be the characteristic functions of the T-games corresponding to the coalitions $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$, respectively.*

For instance, $\upsilon_{\{1,2\}}$ would be:

$$\begin{aligned} \upsilon_{\{1,2\}}(\{1\}) &= 0 \\ \upsilon_{\{1,2\}}(\{2\}) &= 0 \\ \upsilon_{\{1,2\}}(\{3\}) &= 0 \\ \upsilon_{\{1,2\}}(\{1,2\}) &= 1 \\ \upsilon_{\{1,2\}}(\{1,3\}) &= 0 \\ \upsilon_{\{1,2\}}(\{2,3\}) &= 0 \\ \upsilon_{\{1,2\}}(\{1,2,3\}) &= 1 \end{aligned}$$

We can represent the seven different υ values as follows. Note that each one is a 7-tuple.

$$\begin{aligned} \upsilon_{\{1\}} &= (1,0,0,1,1,0,1) \\ \upsilon_{\{2\}} &= (0,1,0,1,0,1,1) \\ \upsilon_{\{3\}} &= (0,0,1,0,1,1,1) \\ \upsilon_{\{1,2\}} &= (0,0,0,1,0,0,1) \\ \upsilon_{\{1,3\}} &= (0,0,0,0,1,0,1) \\ \upsilon_{\{2,3\}} &= (0,0,0,0,0,1,1) \\ \upsilon_{\{1,2,3\}} &= (0,0,0,0,0,0,1) \end{aligned}$$

Lemma 3.1 *Any characteristic function form game can be written as a linear combination of unanimity games:*

$$\upsilon = \sum_{\emptyset \neq T \subseteq N} \lambda_T \upsilon_T$$

where

$$\lambda_T = \sum_{S \subseteq T} (-1)^{|T|-1} \upsilon(T).$$

Since unanimity games are linearly independent, they form a basis for characteristic function form games in Γ (Kóczy [1]).

3.1.2 The Shapley-Value

The Shapley-Value is a solution concept motivated by the need to provide a unique solution for the allocation of expected payments in coalitional games. It was proposed by Shapley in 1953 as part of his doctoral thesis at Princeton University, based on an axiomatic approach. This concept seeks to capture how competitive forces among coalitions affect the potential outcomes of a game. It provides a reasonable or fair method for distributing cooperation gains, considering the strategic realities reflected in the characteristic function of the game. Instead of an explicit formula, The Shapley-value is introduced via three simple axioms (Shapley [5]):

Axiom 3.1 (Symmetry) *For any permutation $\pi \in S_n$ and player $i \in N$,*

$$\varphi(\pi\upsilon) = \pi\varphi(\upsilon)$$

where the game $\pi\upsilon$ is defined as $(\pi\upsilon)(S) = \upsilon(\pi^{-1}(S))\ \forall S \subseteq N$.

This axiom intuitively states that if we rearrange the order of players and their contributions in a cooperative game, The Shapley-value assignments must remain proportionally symmetric to those permutations. In other words, players who are equivalent in terms of their contribution should receive the same payment, regardless of their position in the game.

Axiom 3.2 (Linearity) *Given two coalitional games υ_1 and υ_2 in Γ and any real number α, the following holds:*

$$\varphi(\upsilon_1 + \upsilon_2) = \varphi(\upsilon_1) + \varphi(\upsilon_2) \quad \text{and} \quad \varphi(\alpha\upsilon_1) = \alpha\varphi(\upsilon_1)$$

The linearity axiom shows that in a combined game, the expected payment for each player is a linear combination of their expected payments in the individual games υ_1 and υ_2. In summary, it tells us how The Shapley-value behaves when we linearly combine games, allowing us to calculate The Shapley-value in the combined game as a weighted mixture of The Shapley-values in the original games.

Axiom 3.3 (Carrier) *For any $\upsilon \in \Gamma$ and any carrier coalition $T \subseteq N$ (see Definition 3.2),*

$$\sum_{i \in T} \varphi_i(\upsilon) = \upsilon(N)$$

This axiom states that players in a carrier set must divide the total value they generate together (which is equal to the value of the grand coalition) among themselves. This means that null players receive no allocation since they do not contribute to the overall value.

With the three essential axioms in place, we present the fundamental result by Shapley (3.1).

Theorem 3.1 (Shapley, 1993 [5]) *There exists exactly one assignment, known as the Shapley Value denoted as $Sh : \Gamma \to \mathbb{R}^n$, that satisfies the axioms of linearity, symmetry, and carrier. The formula defining The Shapley-value for player i is as follows:*

$$Sh_i(\upsilon) = \sum_{S \subseteq N^{-i}} \frac{|S|!(n-|S|-1)!}{n!} \{\upsilon(S^{+i}) - \upsilon(S)\} \quad \forall i \in N, \forall \upsilon \in \Gamma \tag{3.1}$$

In this formula:

- The expression $\frac{|S|!(n-|S|-1)!}{n!}$ is interpreted as the probability that, in any permutation, the members of S come before player i.
- The difference $\upsilon(S^{+i}) - \upsilon(S)$ represents the marginal contribution of player i to the value of coalition S.

In summary, The Shapley-value is an assignment that captures the expected contribution of each player to the value of any coalition, satisfying the axioms of linearity, symmetry, and carrier.

Let's consider a collection of n resources, where each resource is uniquely essential to achieving a specific service. Suppose $\upsilon(N)$ represents the total value that this collection of resources would generate if they were all deployed to accomplish the service. Let's focus on a particular resource, let's say resource i. This resource will make a marginal contribution to each subset S of N^{-i} when included in that set. The choice of the set S can be made in $(|S|!(n-|S|-1)!)$ ways, and when divided by $n!$, we obtain the probability of selecting a specific subset S. In this way, The Shapley-value of resource i represents the average marginal contribution that resource i makes to any arbitrary coalition that is a subset of N^{-i}.

There is an *alternative characterization* of the Shapley Value (3.1) if we consider that, the carrier axiom combines two more elementary axioms: Efficiency and the Null player property.

Axiom 3.4 (Null player property) *If $i \in N$ is a null player in $\upsilon \in \Gamma$, then*

$$\varphi_i(\upsilon) = 0 \quad \forall i \notin T$$

Axiom 3.5 (Efficiency) *For the grand coalition N,*

$$\sum_{i \in N} \varphi_i(\upsilon) = \upsilon(N) \quad \forall \upsilon \in \Gamma$$

These axioms emphasize a crucial fact: The Shapley-value always divides the gain of the grand coalition among the players of the game. This reflects the implicit assumption that players are willing to join the grand coalition, even if some of them are null players who do not receive allocations. Furthermore, **these two axioms (along with those of linearity and symmetry) uniquely characterize The Shapley-value**, similar to the concept of carrier. This is relevant as our work is focused in this direction, and understanding the distinctive nature of The Shapley-value was essential to our approach in partition function games. These four axioms lead to the following theorem,

Theorem 3.2 (Alternative characterization of The Shapley-Value) *There exists a unique solution that satisfies the axioms of linearity, symmetry, efficiency, and nullity. Moreover, it this The Shapley-Value.*

Proof. Here we show that there exists exactly one mapping φ that satisfies the four axioms. First, we prove that the mapping φ is a linear transformation by making the following observations,

- Let $z \in \Gamma$ be the coalitional game that assigns worth zero to every coalition, that is, $z(S) = 0$ for all $S \subseteq N$. Then efficiency axiom implies that

$$\varphi_i(z) = 0 \quad \forall i \in N \tag{A1}$$

- From linearity axiom, we have

$$\varphi_i(\alpha\upsilon + (1-\alpha)w) = \alpha\varphi_i(\upsilon) + (1-\alpha)\varphi_i(w), \quad \alpha \in \mathbb{R}, \quad \forall \upsilon, w \in \Gamma$$

  Choosing $w = z$ in the above, we get

$$\varphi_i(\alpha\upsilon) = \alpha\varphi_i(\upsilon) \quad \forall i \in N, \quad \forall \upsilon \in \Gamma \tag{A2}$$

Equations (A1) and (A2) together with the linearity axiom imply that φ is a linear transformation.

Let $T \subseteq N$ be any coalition. Now, note that unanimity game (3.6) implies that a coalition S has worth 1 in υ_T if it contains all the players in T and has worth zero otherwise. To get a feel for this game, we discuss a simple example before resuming the proof.

Then, by the lemma (3.1), for any $\upsilon \in \Gamma$, there exists a unique solution $\lambda_T \in \mathbb{R}$ such that:

$$\upsilon = \sum_{\emptyset \neq T \subseteq N} \lambda_T \cdot \upsilon_T$$

Continuing with the proof, we assume that $\varphi : \Gamma \to \mathbb{R}^n$ is a solution that satisfies all four axioms.

- **By linearity:**

$$\varphi(\upsilon) = \varphi(\sum_{\emptyset \neq T \subseteq N} \lambda_T \cdot \upsilon_T) = \sum_{\emptyset \neq T \subseteq N} \lambda_T \cdot \varphi(\upsilon_T)$$

  Hence, φ will be unique for any $\upsilon \in \Gamma$ if it is also unique for each υ_T with $T \subseteq N$ and $T \neq \emptyset$.

- **By nullity:** If $i \notin T$:

$$\upsilon_T(S^{+i}) = \begin{cases} 1 & \text{if } T \subseteq S^{+i} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} = \upsilon_T(S)$$

  So, if i has a null value in υ_T, then $\varphi_i(\upsilon_T)$ will be equal to zero if i is not in T.

- **By symmetry:** Notice that in the formula for $\varphi_i(\upsilon)$, the only relevant aspects of a coalition are whether it includes player i and the total number of players it contains. In other words, the specific identity of the players within the coalition does not affect the outcome. This observation clearly illustrates the satisfaction of symmetry (Axiom 3.1).
- **By efficiency:** Since $T \subseteq N$, $\upsilon_T(N) = 1$. Then, because of the efficiency of φ, there exists only one unique solution for υ_T,

$$\varphi_i(\upsilon_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$ ■

3.2 Games in Partition Function Form

In games with coalition structures, coalitions are part of the usual practice, and multiple coalitions may (and usually do) coexist. For such games, we must define the coalitional values together.

For a partition $\mathscr{P}$, payoffs to coalitions $C \notin \mathscr{P}$, including the empty set, are conventionally assumed to be undefined, making the definition elegant but somewhat awkward. More contemporary definitions use embedded coalitions:

Definition 3.7 (Partition Function Form Game, Thrall and Lucas, 1963 [6]) *A game in partition function form is a pair (N, w) consisting of a finite set of players N and a partition function*

$$w : \mathscr{E} \to \mathbb{R}$$

that assigns a real value to each embedded coalition $(S,\mathscr{P})$, and such that $w(\emptyset,\mathscr{P}) = 0$ $\forall \mathscr{P}$.

Note that, $\mathscr{E}$ representing the set of *embedded coalitions*, is the set of coalitions together with specifications as to how the other players are aligned.

Definition 3.8 *The set of games in partition function form with player set N is denoted by $\tilde{\Gamma}$, i.e.,*

$$\tilde{\Gamma} = \{w | w : \mathscr{E} \to \mathbb{R} | w(\emptyset, \mathscr{P}) = 0 \ \forall \mathscr{P} \in \tilde{\Pi}(N)\}$$

The value $w(S,\mathscr{P})$ represents the payoff of coalition S, given the coalitional structure $\mathscr{P}$. We can see that in games in partition function form (and that in this work, we will only call them *games*), the worth of some coalition S depends not only on what the players of such coalition can obtain together, but also on the way the other players are organized in $N \setminus S$. We assume that, in any game situation, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N,\{N\})$ to divide among themselves.

Now, given w_1, $w_2 \in \tilde{\Gamma}$ and $\alpha \in \mathbb{R}$, we can define the sum w_1+w_2 and the product αw_1 in $\tilde{\Gamma}$, in the usual form, i.e,

$$(w_1 + w_2)(S,\mathscr{P}) = w_1(S,\mathscr{P}) + w_2(S,\mathscr{P}) \qquad \textit{and} \qquad (\alpha w_1)(S,\mathscr{P}) = \alpha w_1(S,\mathscr{P})$$

respectively. It is easy to verify that $\tilde{\Gamma}$ is a vector space with these operations.

As defined in Γ, a solution is a function $\varphi : \tilde{\Gamma} \to \mathbb{R}^n$. If φ is a solution and $w \in \tilde{\Gamma}$, then we can interpret $\varphi_i(w)$ as the utility payoff which i shlould expect from de game w.

3.2.1 Axioms

We present a list of axioms used in the characterization of extended Shapley values for games in partition function form (Kóczy [1]). The multiplicity of such extensions is, in part, the result of the multiplicity of extensions of axioms for characteristic function form games. Partition function form games are, of course, much more complex, but such a multiplicity of axioms can be confusing, and the different variations may make it more difficult to establish the evidence for these properties. Therefore, we will first focus on the most basic and widely accepted and used axioms and finally examine some alternatives to the null player property.

Axiom 3.6 (Linearity) *The solution φ is linear if*

$$\varphi(w_1 + \alpha w_2) = \varphi(w_1) + \alpha\varphi(w_2) \quad \forall w_1, w_2 \in \tilde{\Gamma} \text{ and } \alpha \in \mathbb{R}$$

Axiom 3.7 (Symmetry) *The solution φ is said to be symetric if and only if $\varphi(\pi w) = \pi\varphi(w)$ $\forall \pi \in S_n$, where the game πw is defined as*

$$(\pi w)(S,\mathscr{P}) = w[\pi^{-1}(S,\mathscr{P})], \quad \forall (S,\mathscr{P}) \in \mathscr{E}$$

The interpretation of these axioms is analogous to the axioms used in solutions for traditional games.

Myerson's article in 1977 represented a significant achievement by proposing a method for allocating fair contributions in cooperative games (Myerson [3]). This method is based on three fundamental axioms, analogous to Shapley but applied to games expressed in the form of a partition function: linearity, symmetry, and the ***carrier axiom***, which encompasses efficiency and nullity in this context.

Definition 3.9 (Carrier in Partition Function Form Games, Myerson [3]) *Given $w \in \tilde{\Gamma}$, the set T is a carrier of w if for any embedded coalition $(S,\mathscr{P})$,*

$$w(S,\mathscr{P}) = w(S \cap T, \mathscr{P} \wedge \{T, \overline{T}\}) \tag{3.2}$$

In other words, T is considered a carrier of w if the outcomes obtained by players in S when cooperating in coalition $\mathscr{P}$ are the same as what they would achieve when restricted to T. It's worth noting that N always acts as a carrier since if a player i is not in T, then $w(\{i\},\mathscr{P}) = w(\{i\} \cap T, \mathscr{P}) = w(\emptyset,\mathscr{P}) = 0$, i.e., if a player has no influence on the outcome, they should neither receive nor pay anything.

The following axiom suggests that the total amount obtained by the grand coalition should be distributed among the members of a carrier:

Axiom 3.8 (Carrier) *For any $w \in \tilde{\Gamma}$, if T is a carrier of w, then the following holds*

$$\sum_{i \in T} \varphi_i(w) = w(N, \{N\})$$

Myerson proceeds axiomatically similarly to Shapley and proposes a value that extends the Shapley-value. His proposal satisfies the axioms of linearity, symmetry, and carrier. The Myerson value for a player is given by:

Theorem 3.3 (Myerson [3]) *The solution $My : \tilde{\Gamma} \to \mathbb{R}^n$ defined as follows:*

$$My_i(w) = \sum_{(S,\mathscr{P})\in\mathscr{E}} (-1)^{|\mathscr{P}|-1}(|\mathscr{P}|-1)!\left[\frac{1}{n} - \sum_{\substack{T\in\mathscr{P}\setminus\{S\}\\ i\notin T}} \frac{1}{(|\mathscr{P}|-1)(n-|T|)}\right]\cdot w(S,\mathscr{P}) \tag{3.3}$$

where $i \in N$ and $w \in \tilde{\Gamma}$, represents the unique solution that satisfies the axioms of *Linearity*, *Symmetry*, and *Carrier*.

Example 3.2 *Now, if $N = \{1,2,3\}$, according to the Myerson value, the payoff for player 1 is*

$$\begin{aligned} My_1(w) =& \frac{1}{3}\cdot w(\{1,2,3\},\{\{1,2,3\}\}) \\ &-\frac{1}{6}\cdot(2w(\{1\},\{\{1\},\{2\},\{3\}\}) - w(\{2\},\{\{1\},\{2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\})) \\ &+\frac{1}{3}\cdot(2w(\{1\},\{\{1\},\{2,3\}\}) - w(\{2\},\{\{2\},\{1,3\}\}) - w(\{3\},\{\{3\},\{1,2\}\})) \\ &+\frac{1}{6}\cdot(w(\{1,2\},\{\{3\},\{1,2\}\}) + w(\{1,3\},\{\{2\},\{1,3\}\}) - 2w(\{2,3\},\{\{1\},\{2,3\}\})) \end{aligned}$$

In a corollary, Myerson demonstrates that his value is a consistent extension of the Shapley-value:

Corollary 3.3.1 (Myerson [3]) *Suppose $w \in \tilde{\Gamma}$ and $\upsilon \in \Gamma$ satisfy $w(S,\mathscr{P}) = \upsilon(S), \forall (S,\mathscr{P}) \in \mathscr{E}$. Then,*

$$Sh(\upsilon) = My(w)$$

This corollary establishes a relationship between two functions that assign values to coalitions in the context of cooperative games and is related to the concept of the *Shapley value*:

- First, we consider a function $w \in \tilde{\Gamma}$ that assigns values to coalitions in a cooperative game. This means that $w(S,\mathscr{P})$ represents the value that the game assigns to coalition S within the partition $\mathscr{P}$.
- Next, we have another function $\upsilon \in \Gamma$ that also assigns values to coalitions, albeit in a different manner compared to w. In this case, $\upsilon(S)$ indicates the value assigned by υ to coalition S.

The corollary states that if these two functions, w and υ, satisfy the equality $w(S,\mathscr{P}) = \upsilon(S)$ for all coalitions in the set $\mathscr{E}$, then the Myerson-value of w is equal to the Shapley-value of υ.

In a corollary, Myerson posits that a stronger version of the carrier axiom (3.8). Given $\mathscr{P} \in \Pi$ and $w \in \tilde{\Gamma}$, it is asserted that w is *$\mathscr{P}$-decomposable* iff $\forall (T,\mathscr{Q})$,

$$w(T,\mathscr{Q}) = \sum_{S\in\mathscr{P}} w(T\cap S, \mathscr{Q}\wedge\mathscr{P}) \tag{3.4}$$

(Recall $w(\emptyset,\mathscr{P})$=0).

This equality as a whole signifies that the total value assigned by the function w to the coalition T in the context of the partition $\mathscr{Q}$ is equal to the sum of the individual contributions of the subsets S that belong to the partition $\mathscr{P}$. Each individual contribution is calculated by considering the intersection of the players present in both T and S, in the context of the new partition $\mathscr{Q} \wedge \mathscr{P}$. This implies that the total value that T receives in the context of $\mathscr{Q}$ is the sum of what each subset S contributes to the intermediate partition $\mathscr{Q} \wedge \mathscr{P}$.

In summary, this equality demonstrates how the value of a coalition T is distributed based on how subsets S contribute through the intermediate partition $\mathscr{Q} \wedge \mathscr{P}$, taking the intersection of players into account in the process.

Corollary 3.3.2 (Myerson [3]) *If $w \in \tilde{\Gamma}$ is $\mathscr{P}-$decomposable, then*

$$\sum_{i \in S} My_i(w) = w(S,\mathscr{P}), \quad \text{for any } S \in \mathscr{P}$$

The carrier axiom establishes that the sum of the individual values of players in any coalition S acting as a carrier is equal to the total value of the grand coalition, i.e., $w(N,\{N\}) = w(S,\{S,\overline{S}\})$, regardless of the involved function w. Let's consider a partition $\mathscr{P}$ containing only the grand coalition N, i.e., $\mathscr{P} = \{N\}$. By applying Corollary 3.3.2 and the Carrier Axiom 3.8, we arrive at the following conclusion:

$$\sum_{i \in N} \varphi_i(w) = w(N,\mathscr{P}) = w(N,\{N\})$$

Given that both Corollary 3.3.2 and the carrier axiom apply to any game function w in the set Γ, we can conclude that the efficiency axiom holds for any game function $w \in \tilde{\Gamma}$. This demonstrates that the sum of the individual values of the players in the grand coalition, acting as a carrier, is equal to the total value of the grand coalition. This leads us to establish the following axiom:

Axiom 3.9 (Efficiency) *For any partition function form game (N,w),*

$$\sum_{i \in N} \varphi_i(w) = w(N,\{N\}) \quad \forall w \in \tilde{\Gamma}. \tag{3.5}$$

3.2.2 LSE solutions and the Transfers Procedure

Here we present a family of solutions that satisfy the axioms of linearity, symmetry, and efficiency establishing the fundamental groundwork for our work.

Definition 3.10 (LSE solution) *A solution that satisfies linearity, symmetry and efficiency axioms is referred to as an LSE solution.*

Definition 3.11 *Let B_n be a set of triples, associated with all partitions $|S| \in \lambda^\circ$ and its elements, i.e.,*

$$B_n = \{(\lambda,|S|,|T|) | \lambda \in \Lambda(n) \setminus \{[n]\}, |S| \in \lambda^\circ, |T| \in (\lambda - [|S|])^\circ\}$$

Example 3.3 *If $n = 4$, then*

$$\begin{aligned} B_4 = \{&([1111],1,1),([211],1,1),([211],1,2),([211],2,1) \\ &([22],2,2),([31],1,3),([31],3,1)\} \end{aligned}$$

Theorem 3.4 (Hernández-Lamoneda, et al. [7]) *The solution $\psi : \tilde{\Gamma} \to \mathbb{R}^n$ satisfies linearty, symmetry and efficiency axioms if only if it is of the form*

$$\psi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{(\lambda,|S|,|T|)\in B_n} \beta_{(\lambda,|S|,|T|)} \times \Big[\sum_{\substack{(S,\mathscr{P})\in\mathscr{E} \\ \lambda_{\mathscr{P}}=\lambda, S\ni i}} \sum_{T\in\mathscr{P}\setminus\{S\}} |T| w(S,\mathscr{P}) - \sum_{\substack{(S,\mathscr{P})\in\mathscr{E} \\ \lambda_{\mathscr{P}}=\lambda, S\not\ni i \\ |\mathscr{P}(i)|=|T|}} |S| w(S,\mathscr{P}) \Big] \tag{3.6}$$

for some real numbers $\{\beta_{(\lambda,|S|,|T|)} | (\lambda,|S|,|T|) \in B_n\}$.

Even though the parameters β in equation (3.6) can take any real number values, when $\beta_{(\lambda,|S|,|T|)} \in [0,1]$, we can interpret them as weights or fractions of the wealth $w(S,\mathscr{P})$.

Now, we describe the final payoff for player $i \in N$ as a result of the following elementary procedure:

1. He receives the egalitarian amount $\dfrac{w(N,\{N\})}{n}$.
2. For each embedded coalition $(S,\mathscr{P})$ such that $S \neq N$, there are transfers between players in S and players in $\overline{S}$.
   - If player i belongs to S, then he receives (from each player in each $T \in \mathscr{P} \setminus \{S\}$) a fraction $\beta_{(\lambda_{\mathscr{P}},|S|,|T|)}$ of the worth $w(S,\mathscr{P})$. In total from coalition T:
   $$|T| \cdot \beta_{(\lambda_{\mathscr{P}},|S|,|T|)} \cdot w(S,\mathscr{P})$$
   - If player i does not belong to S, then he pays (to each player in S) a fraction $\beta_{(\lambda_{\mathscr{P}},|S|,|\mathscr{P}(i)|)}$ of the worth $w(S,\mathscr{P})$. In total to coalition S:
   $$|S| \cdot \beta_{(\lambda_{\mathscr{P}},|S|,|\mathscr{P}(i)|)} \cdot w(S,\mathscr{P})$$

Notice that the weights are symmetric in the following sense:

- If $i \in S$, then the weights associated to the embedded coalition $(S,\mathscr{P})$ depend on three parameters: the structure of $\mathscr{P}$, the cardinality of S and the cardinality of other coalition T different from S.
- In $i \notin S$, the weights depend on: the structure of $\mathscr{P}$, the cardinality od S and the cardinality of the coalition that contains player i.

Example 3.4 *If $N = \{1,2,3\}$, then every LSE solution is of the form (for player 1):*

$$\begin{aligned}\psi_1(w) = &\frac{1}{3} \cdot w(\{1,2,3\},\{\{1,2,3\}\}) + \beta_{([111],1,1)} \cdot \big[2w(\{1\},\{\{1\},\{2\},\{3\}\}) \\ &-w(\{2\},\{\{1\},\{2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\})\big] \\ &+\beta_{([21],1,2)} \cdot \big[2w(\{1\},\{\{1\},\{2,3\}\}) - w(\{2\},\{\{2\},\{1,3\}\}) - w(\{3\},\{\{3\},\{1,2\}\})\big] \\ &+\beta_{([21],2,1)} \cdot \big[w(\{1,2\},\{\{3\},\{1,2\}\}) + w(\{1,3\},\{\{2\},\{1,3\}\}) \\ &-2w(\{2,3\},\{\{1\},\{2,3\}\})\big]\end{aligned}$$

where $\beta_{([111]1,1)}$, $\beta_{([21]1,2)}$ and $\beta_{([21]2,1)}$ are arbitrary real numbers.

3.2.3 Examples of LSE solutions

In this section we briefly present some solutions that can be imple-mented from the transfers procedure; i.e., they all are of the form (3.6).

As a first example, we take the expected stand-alone value, ψ^{ESA},which tells us how much a player may obtain in a game with externalities when we focus on the stand-alone side of the game:

$$\psi_i^{ESA}(w) = \frac{w(N,\{N\})}{n} + \sum_{\substack{\emptyset \neq S \subset N \\ i \notin S}} \frac{|S|!(n-|S|-1)!}{n!} w(\{i\},\{S\} \cup [N \setminus (S \cup \{i\})] \cup \{\{i\}\}) - \sum_{j \in N^{-i}} \sum_{S \subset N \setminus \{i,j\}} \frac{|S|!(n-|S|-2)!}{n!} w(\{j\},[N \setminus (S \cup \{i\})] \cup \{S \cup \{i\}\})$$

which we obtain when we take in (3.6):

$$\beta_{(\lambda,|S|,|T|)} = \begin{cases} \frac{(|S|-1)!(n-|S|-2)!}{n!} & \text{if } \lambda \in \{[m,1,...,1]\}_{n-1}^{m=1}, |S| = 1 \text{ and } |T| = m \\ 0 & \textit{otherwise} \end{cases}$$

Shapley Value

(Pham Do and Norde [8]) define an extension of the Shapley (Shapley [5]) value to the class of games in partition function form as

$$\psi_i(w) = Sh_i(v)$$

for each $i \in N$ and each $w \in \tilde{\Gamma}$, where $Sh : \Gamma \to \mathbb{R}^n$ is the Shapley value operator for TU games and $\upsilon \in \Gamma$ is defined as follows:

$$\upsilon(S) = w(S, \{S, [N \setminus S]\})$$

for each $S \subseteq N$.

This solution is of the form (3.6) with parameters

$$\beta_{(\lambda,|S|,|T|)} = \begin{cases} \frac{(|S|-1)!(n-|S|-1)!}{n!} & \text{if } \lambda \in \{[m,1,...,1]\}_{n-1}^{m=1}, |S| = m \text{ and } |T| = 1 \\ 0 & \textit{otherwise}. \end{cases}$$

Consensus value

(Ju [9]) defines the consensus value, ϕ_J, as the middle point between the stand-alone value and the Shapley value of (Pham Do and Norde [8]). The corresponding parameters for the consensus value are:

$$\beta_{(\lambda,|S|,|T|)} = \begin{cases} \frac{1}{2n(n-2)} & \text{if } \lambda = [1,1,\dots,1] \text{ and } |S| = |T| = 1 \\ \frac{1}{2n(n-1)(n-2)} & \text{if } \lambda \in \{[m,1,\dots,1]\}_{n-1}^{m=1}, |S| = 1 \text{ and } |T| = m \\ \frac{(|S|-1)!(n-|S|-1)!}{2n!} & \text{if } \lambda \in \{[m,1,\dots,1]\}_{n-1}^{m=1}, |S| = m \text{ and } |T| = 1 \\ 0 & \textit{otherwise}. \end{cases}$$

The value of Albizuri et al.

(Albizu et al. [10]) obtain a unique value characterized by the properties of linearity, symmetry, efficiency, oligarchy, and an additional symmetry requirement with respect to the embedded coalitions. They define the value for a player as:

$$\psi_i^{AAR}(w) = \sum_{\substack{(S,\mathscr{P})\in\mathscr{E} \\ i\in S}} \frac{(|S|-1)!(n-|S|)!}{n!P(S,N)} w(S,\mathscr{P}) - \sum_{\substack{(S,\mathscr{P})\in\mathscr{E} \\ i\in S}} \frac{|S|!(n-|S|-1)!}{n!P(S,N)} w(S,\mathscr{P})$$

where

$$P(S,N) = |\{(T,\mathscr{P}) \in \mathscr{E} \mid T = S\}|$$

In fact, they notice that $P(S,N) = p(n-|S|)$, where $p(k)$ represents the number of partitions of any set K with cardinality k. This solution is also of the form (3.6). The corresponding parameters are:

$$\beta_{(\lambda,|S|,|T|)} = \frac{(n-|S|-1)!(|S|-1)!}{n! \cdot p(n-|S|)}$$

The value of Macho-Stadler et al.

As a final example, (Macho-Stadler et al. [11]) characterize the value:

$$\psi_i^{MPW}(w) = \sum_{\substack{(S,\mathscr{P})\in\mathscr{E} \\ i\in S}} \frac{(|S|-1)! \prod_{T\in\mathscr{P}\setminus\{S\}} (|T|-1)!}{n!} w(S,\mathscr{P}) - \sum_{\substack{(S,\mathscr{P})\in\mathscr{E} \\ i\notin S}} \frac{|S|! \prod_{T\in\mathscr{P}\setminus\{S\}} (|T|-1)!}{(n-|S|)n!} w(S,\mathscr{P})$$

which we get when we choose:

$$\beta_{(\lambda,|S|,|T|)} = \frac{(|S|-1)! \prod_{\delta\in(\lambda-[|S|])^\circ} [(\delta-1)!]^{m_\delta^{\lambda-[|S|]}}}{(n-|S|)n!}$$

Example 3.5 *We look at the system of weights for the different solutions described above in the case n= 3. For this particular case,there are 3 different weights associated with the transfers procedure:*

Table 3.1 *System of weights for LSE solutions for n = 3.*

Weights	ψ^{ESA}	ψ^{PN}	ψ^{J}	ψ^{M}	ψ^{AAR}	ψ^{MPW}
$\beta_{([111],1,1)}$	$^1/_6$	$^1/_6$	$^1/_6$	$-^1/_6$	$^1/_{12}$	$^1/_{12}$
$\beta_{([21],1,2)}$	$^1/_6$	0	$^1/_{12}$	$^1/_3$	$^1/_{12}$	$^1/_{12}$
$\beta_{([21],2,1)}$	0	$^1/_6$	$^1/_{12}$	$^1/_6$	$^1/_6$	$^1/_6$

Now, for the case n = 4 there are 7 different weights associated with the transfers procedure (table 3.2):

Table 3.2 *System of weights for LSE solutions for n = 4.*

Weights	ψ^{ESA}	ψ^{PN}	ψ^{J}	ψ^{M}	ψ^{AAR}	ψ^{MPW}
$\beta_{([1111],1,1)}$	$^1/_{24}$	$^1/_{12}$	$^1/_{16}$	$^1/_6$	$^1/_{60}$	$^1/_{72}$
$\beta_{([211],2,1)}$	0	$^1/_{24}$	$^1/_{48}$	$-^1/_{12}$	$^1/_{48}$	$^1/_{48}$
$\beta_{([211],1,2)}$	$^1/_{24}$	0	$^1/_{48}$	$-^1/_6$	$^1/_{60}$	$^1/_{72}$
$\beta_{([211],1,1)}$	0	0	0	0	$^1/_{60}$	$^1/_{72}$
$\beta_{([31],3,1)}$	0	$^1/_{12}$	$^1/_{24}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$
$\beta_{([31],1,3)}$	$^1/_{24}$	0	$^1/_{48}$	$^1/_4$	$^1/_{60}$	$^1/_{36}$
$\beta_{([22],2,2)}$	0	0	0	$^1/_8$	$^1/_{48}$	$^1/_{48}$

As we can observe, in both cases, the Myerson value is the only solution that considers negative weights.

4 Development and results

"There is no other economic principle that is so fatal as that of trying to make people happier."

FRIEDRICH HAYEK

In the upcoming chapter, we will introduce an innovative version of the null player concept and explore its application in LSE solutions. Throughout this exploration, we will encounter an intriguing observation: the resulting solutions are infinite, contingent on various arbitrary parameters. By excluding externalities in these partition function solutions, we derive the well-known Shapley value.

Subsequently, we will proceed to characterize the diverse solution families stemming from this null player proposal. Finally, a detailed comparison will be conducted with other existing definitions and proposals of the null player. This comprehensive approach aims to provide a deeper understanding of the complexities and implications associated with this novel perspective in the realm of cooperative games.

4.1 Null Players in Partition Function Form Games: A Novel Proposal

The purpose of this section is to ascertain whether it is possible to uniquely characterize the Myerson value based on a null player axiom and an efficiency axiom, similar to the approach used for the Shapley value. The conditions/restrictions derived from the null player axiom will be applied to the expression (3.6) of the linear, symmetric, and efficient solutions. We will follow the same line of reasoning as before: *A player $i \in N$ is deemed null in $w \in \tilde{\Gamma}$ if only if N^{-i} is a carrier of w.*

According to the definition above, if we have a partition $\mathscr{P} = \{S_1, S_2, \ldots, \mathscr{P}(i), \ldots, S_m\}$ of N, we can observe the following:

1. If $i \in S$, then:

$$\begin{aligned} w(S,\mathscr{P}) &= w(S \cap N^{-i}, \mathscr{P} \wedge \{N^{-i}, N \setminus N^{-i}\}) \\ &= w(S^{-i}, \mathscr{P} \wedge \{N^{-i}, \{i\}\}) \\ &= w(S^{-i}, \{S_1, S_2, \ldots, (S^{-i}, \{i\}, \ldots, S_m\}) \\ &= w(S^{-i}, \{S^{-i}, \{i\}\} \cup \mathscr{P} \setminus \{S\}) \end{aligned}$$

2. If $i \notin S$, then:

$$\begin{aligned} w(S,\mathscr{P}) &= w(S \cap N^{-i}, \mathscr{P} \wedge \{N^{-i}, N \setminus N^{-i}\}) \\ &= w(S, \mathscr{P} \wedge \{N^{-i}, \{i\}\}) \\ &= w(S, \{S_1, S_2, ..., (\mathscr{P}(i))^{-i}, \{i\}, ..., S_m\}) \\ &= w(S, \{(\mathscr{P}(i))^{-i}, \{i\}\} \cup \mathscr{P} \setminus \{\mathscr{P}(i)\}) \end{aligned}$$

If $i \notin S$, then $S = S^{-i}$. This leads us to the following definition

Definition 4.1 (Null Player in Partition Function Form) *Given a partition function form game (N,w), a player $i \in N$ is a null player in $w \in \tilde{\Gamma}$ if $\forall (S,\mathscr{P}) \in \mathscr{E}$,*

$$w(S,\mathscr{P}) = w(S^{-i},\{(\mathscr{P}(i))^{-i},\{i\}\} \cup \mathscr{P} \setminus \{\mathscr{P}(i)\}) \tag{4.1}$$

This proposal evaluates how the wealth generated by the coalition S varies depending on whether player i is part of that coalition or not. To do this, it adjusts the partition $\mathscr{P}$ based on whether i is present or not in the involved coalitions. Removing player i from all coalitions reflects the question of how wealth would be distributed if i were not present. For instance, if n=4, $\mathscr{P} = \{\{1,2\},\{3,4\}\}$, and player 1 is null, then $w(\{1,2\},\{\{1,2\},\{3,4\}\})$ = $w(\{2\},\{\{1\},\{2\},\{3,4\}\})$ y $w(\{3,4\},\{\{1,2\},\{3,4\}\})$ = $w(\{3,4\},\{\{1\},\{2\},\{3,4\}\})$.

Now we are ready to state the following axiom:

Axiom 4.1 (Nullity) *If $i \in N$ is a null player in $w \in \tilde{\Gamma}$, then $\psi_i(w) = 0$, $\forall w \in \tilde{\Gamma}$.*

Next, we present the application of the equation (3.6) to a game with three agents. In this context, we apply axiom 4.1 to identify a null player and *Definition 4.1* (*Null Player in Partition Function Form*) to assess how wealth is distributed when the null player is present or absent in the coalitions. This leads to the following expression for the payoff of player 1 (Example 4.1).

Example 4.1 *From (3.6), the LSE solution satisifes the nullity axiom if and only if $\beta_{([21],2,1)} = {}^{1}/_{6}$ and $\beta_{([111],1,1)} + \beta_{([21],1,2)} = {}^{1}/_{6}$.*

Let $\beta_{([111],1,1)} = \beta$; thus:

$$\begin{aligned} \psi_1(w) = \frac{1}{3} \cdot w(\{1,2,3\},\{\{1,2,3\}\}) + \beta \cdot \big[2w(\{1\},\{\{1\},\{2\},\{3\}\}) \\ - w(\{2\},\{\{1\},\{2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\}) - 2w(\{1\},\{\{1\},\{2,3\}\}) \\ + w(\{2\},\{\{2\},\{1,3\}\}) + w(\{3\},\{\{3\},\{1,2\}\})\big] \\ + \frac{1}{6} \cdot \big[2w(\{1\},\{\{1\},\{2,3\}\}) - 2w(\{2,3\},\{\{1\},\{2,3\}\}) + w(\{1,2\},\{\{3\},\{1,2\}\}) \\ - w(\{3\},\{\{3\},\{1,2\}\}) + w(\{1,3\},\{\{2\},\{1,3\}\}) - w(\{2\},\{\{2\},\{1,3\}\})\big] \end{aligned}$$

It can be observed that, since $\beta \in \mathbb{R}$ is arbitrary, there is an entire class of solutions that fulfill the four axioms, with the Myerson value being a particular case for $\beta = -{}^{1}/_{6}$.

Example 4.2 *If we assume that there are no externalities, i.e.,*

$$\begin{aligned} w(\{1\},\{\{1\},\{2\},\{3\}\}) &= w(\{1\}\{\{1\},\{2,3\}\}) = w(\{1\}) \\ w(\{2\},\{\{1\},\{2\},\{3\}\}) &= w(\{2\},\{\{2\},\{1,3\}\}) = w(\{2\}) \\ w(\{3\},\{\{1\},\{2\},\{3\}\}) &= w(\{3\},\{\{3\},\{1,2\}\}) = w(\{3\}) \end{aligned}$$

we have

$$\begin{aligned} \psi_1(w) = \frac{w(\{1,2,3\})}{3} + \frac{1}{6} \cdot [&2(w(\{1\}) - w(\{2,3\})) + w(\{1,2\}) - w(\{3\}) \\ &+ w(\{1,3\}) - w(\{2\})] \end{aligned} \tag{4.2}$$

that is the Shapley-value for player 1. Similarly, the process is carried out for the payments of players 2 and 3.

If we consider the following

$\mathscr{P}$	**Coalitions**	**Worth**
1	{1}, {2}, {3}	(1,4,2)
2	{1,2}, {3}	(6,2)
3	{1,3}, {2}	(4,4)
4	{2,3}, {1}	(8,1)
5	{1,2,3}	(8)

equations (3.1) and (4.2) give us the same value:

$$Sh(\upsilon) = \psi(w) = \left(1, \frac{9}{2}, \frac{5}{2}\right)$$

Example 4.3 *If we proceed in a similar way to the previous example, but for 4 agents, we have:*

$$\begin{aligned} \psi_1(w) = \frac{w(\{1,2,3,4\})}{4} + \frac{1}{12} \cdot [&3(w(\{1\}) - w(\{1,3,4\})) + w(\{1,2,3\}) - w(\{4\}) \\ &+ w(\{1,2,4\}) - w(\{3\}) + w(\{1,3,4\}) - w(\{2\}) \\ &+ w(\{1,2\}) - w(\{3,4\}) + w(\{1,3\}) - w(\{2,4\}) \\ &+ w(\{1,4\}) - w(\{2,3\})] \end{aligned} \tag{4.3}$$

with,

$\mathscr{P}$	Coalitions	Worth	$\mathscr{P}$	Coalitions	Worth
1	{1}, {2}, {3}, {4}	(10,10,20,30)	9	{1,3}, {2,4}	(30,40)
2	{1,2}, {3}, {4}	(18,20,30)	10	{1,4}, {2,3}	(40,26)
3	{1,3}, {2}, {4}	(30,10,30)	11	{1,2,3}, {4}	(32,30)
4	{1,4}, {2}, {3}	(40,10,20)	12	{1,2,4}, {3}	(48,20)
5	{2,3}, {1}, {4}	(26,10,30)	13	{1,3,4}, {2}	(52,10)
6	{2,4}, {1}, {3}	(40,10,20)	14	{2,3,4}, {1}	(52,10)
7	{3,4}, {1}, {2}	(44,10,10)	15	{1,2,3,4}	(60)
8	{1,2}, {3,4}	(18,44)			

then,

$$\psi(w) = Sh(\upsilon) = \left(\frac{26}{3}, 8, \frac{46}{3}, 28\right)$$

that is the Shapley-value.

4.2 Characterization of Families of Satisfactory Solutions under the LSEN Approach

In this section, we delve into the detailed characterization of the LSEN (Linear, Symmetric, Efficient, and Null) family of solutions previously introduced. Before delving into the exploration and analysis of these solutions, we will begin by establishing and defining fundamental concepts that will be essential for their understanding.

Let F_n be a set of pairs:

$$F_n = \{(\lambda, |S|) | \lambda \in \Lambda(n), |S| \in \lambda^\circ \setminus \{1, n\}\}$$

and, for $\lambda \in \Lambda(n)$, $\delta \neq 1$ and $\delta, \varepsilon \in \lambda$, we define,

$$\lambda_\varepsilon^\delta = \lambda - [\delta, \varepsilon] + [\varepsilon + 1, \delta - 1].$$

Example 4.4 *If $n = 5$, then*

$$F_5 = \{(2111,2),(221,2),(311,3),(32,2),(32,3),(41,4)\}$$

so, for the pair $(2111,2)$, we have $m_2^\lambda = 1$. Additionally, for $\varepsilon \neq \delta$ with $\varepsilon \in \lambda^\circ \cup \{0\}$, we obtain $\lambda_0^2 = [11111]$ and $\lambda_1^2 = [2111]$.

Now, we propose the following expression for the resulting equations when applying the nullity axiom to the LSEN solution:

Theorem 4.1 *The solution $\psi : \tilde{\Gamma} \to \mathbb{R}^n$ satisfies linearty, symmetry, efficiency and nullity axioms if only if it is of the form*

$$\begin{aligned} \psi_i(w) =& \frac{w(N, \{N\})}{n} + \sum_{(\lambda, |S|, |T|) \in B_n} \beta_{(\lambda, |S|, |T|)} \\ & \times \Bigg[\sum_{\substack{(S, \mathscr{P}) \in \mathscr{E} \\ \lambda_{\mathscr{P}} = \lambda, S \ni i}} \sum_{T \in \mathscr{P} \setminus \{S\}} |T| w(S, \mathscr{P}) - \sum_{\substack{(S, \mathscr{P}) \in \mathscr{E} \\ \lambda_{\mathscr{P}} = \lambda, S \not\ni i \\ |\mathscr{P}(i)| = |T|}} |S| w(S, \mathscr{P}) \Bigg] \end{aligned} \tag{4.4}$$

for some real numbers $\{\beta_{(\lambda,|S|,|T|)}|(\lambda,|S|,|T|)\in B_n\}$ *such that*

i)

$$\beta_{([n-1,1],n-1,1)}=\frac{1}{n(n-1)} \tag{4.5}$$

and

ii) *For every* $(\lambda,\delta)\in F_n$*:*

$$\begin{aligned}&\left(m^{\lambda}_{\delta}-1\right)\left[\delta\beta_{(\lambda,\delta,\delta)}-(\delta-1)\beta_{(\lambda^{\delta}_{\delta},\delta-1,\delta+1)}\right]+\\&\sum_{\substack{\varepsilon\in\lambda^{\circ}\cup\{0\}\\ \varepsilon\neq\delta}}\left[\varepsilon m^{\lambda}_{\varepsilon}\beta_{(\lambda,\delta,\varepsilon)}-(\delta-1)m^{\lambda}_{\varepsilon}\beta_{(\lambda^{\delta}_{\varepsilon},\delta-1,\varepsilon+1)}\right]=0\end{aligned} \tag{4.6}$$

Moreover, such representation is unique.

Proof. (⇒) From Hernández-Lamoneda et al. ([7], Theorem 4),

$$\begin{aligned}\psi_i(w)=&\frac{w(N,\{N\})}{n}+\sum_{(\lambda,|S|,|T|)\in B_n}\beta_{(\lambda,|S|,|T|)}\\&\times\Big[\sum_{\substack{(S,\mathscr{P})\in\mathscr{E}\\ \lambda_{\mathscr{P}}=\lambda,S\ni i}}\sum_{T\in\mathscr{P}\setminus\{S\}}|T|w(S,\mathscr{P})-\sum_{\substack{(S,\mathscr{P})\in\mathscr{E}\\ \lambda_{\mathscr{P}}=\lambda,S\not\ni i\\ |\mathscr{P}(i)|=|T|}}|S|w(S,\mathscr{P})\Big]\end{aligned} \tag{4.7}$$

is a linear, symmetric and efficient solution for arbitrary constants $\{\beta_{(\lambda,|S|,|T|)}|(\lambda,|S|,|T|)\in B_n\}$. Now, consider the collection of games $\{u_{(S,\mathscr{P})}\mid(S,\mathscr{P})\in\mathscr{E},S\neq\emptyset\}$ defined by

$$u_{(S,\mathscr{P})}(T,\mathscr{Q})=\begin{cases}1 & \text{if } (T,\mathscr{Q})=(S,\mathscr{P})\\ 0 & \text{otherwise}\end{cases}$$

which constitutes a basis for $\widetilde{\Gamma}$ (notice that the dimension of $\widetilde{\Gamma}$ equals the number of nontrivial embedded coalitons). Suppose $i\in N$ is a null player in $u_{(S,\mathscr{P})}$ for every $(S,\mathscr{P})\in\mathscr{E}$; that is, $u_{(S,\mathscr{P})}(R,\mathscr{Q})=u_{(S,\mathscr{P})}\left(R^{-i},\left\{(\mathscr{Q}(i))^{-i},\{i\}\right\}\cup\mathscr{Q}\setminus\mathscr{Q}(i)\right)$ for each $(R,\mathscr{Q})\in\mathscr{E}$.

Nullity implies:

1.

$$0=\psi_i(u_{(N,\{N\})})=\frac{1}{n}-(n-1)\beta_{([n-1,1],n-1,1)}$$

Hence,

$$\beta_{([n-1,1],n-1,1)}=\frac{1}{n(n-1)}$$

2.

$$0=\psi_i(u_{(S,\mathscr{P})})=\sum_{T\in\mathscr{P}\setminus S}\left[|T|\beta_{(\lambda_{\mathscr{P}},|S|,|T|)}-(|S|-1)\,\beta_{(\{\mathscr{P}^i(T),T^{+i}\}\cup\mathscr{P}\setminus\{S,T\},|S|-1,|T|+1)}\right] \tag{4.8}$$

for every pair $(S,\mathscr{P})$ such that $|S|\notin\{1,n\}$ and $i\in S$. Notice that the above relation yields many repeated equations. In particular, relation (4.8) provides the same equation for $(S,\mathscr{P})$ and $(S',\mathscr{P}')$, if $|S|=|S|'$ and $\lambda_{\mathscr{P}}=\lambda_{\mathscr{P}'}$. Thus, the number of distinct equations derived from (4.8) coincides with the number of elements in F_n.

On the other hand, for a fixed $(S,\mathscr{P})$ such that $|S|\notin\{1,n\}$, it holds:

$$\sum_{T\in\mathscr{P}\setminus S}|T|\beta_{(\lambda_{\mathscr{P}},|S|,|T|)}=\begin{cases}\left(m_{|S|}^{\lambda_{\mathscr{P}}}-1\right)|S|\beta_{(\lambda_{\mathscr{P}},|S|,|S|)} & \text{if } |T|=|S|\\ \sum\limits_{\substack{\varepsilon\in\lambda_{\mathscr{P}}^{\circ}\cup\{0\}\\ \varepsilon\neq|S|}}\varepsilon m_{\varepsilon}^{\lambda_{\mathscr{P}}}\beta_{(\lambda_{\mathscr{P}},|S|,\varepsilon)} & \text{if } |T|\neq|S|\end{cases}\tag{4.9}$$

and

$$\sum_{T\in\mathscr{P}\setminus S}\beta_{(\{\mathscr{P}^i(T),T^{+i}\}\cup\mathscr{P}\setminus\{S,T\},|S|-1,|T|+1)}=\begin{cases}\left(m_{|S|}^{\lambda_{\mathscr{P}}}-1\right)\beta_{\left((\lambda_{\mathscr{P}})_{|S|}^{|S|},|S|-1,|S|+1\right)}\\ \text{if } |T|=|S|\\ \sum\limits_{\substack{\varepsilon\in\lambda_{\mathscr{P}}^{\circ}\cup\{0\}\\ \varepsilon\neq|S|}} m_{\varepsilon}^{\lambda_{\mathscr{P}}}\beta_{\left((\lambda_{\mathscr{P}})_{\varepsilon}^{|S|},|S|-1,\varepsilon+1\right)}\\ \text{if } |T|\neq|S|\end{cases}\tag{4.10}$$

The system (4.6) follows from the substitution of equalities (4.9) and (4.10) in relation (4.8).

(⇐) The converse is a straightforward computation in view of the equalities in part (⇒). Finally, to check uniqueness it is enough to prove that if

$$\begin{aligned}0=&\frac{w(N,\{N\})}{n}+\sum_{(\lambda,|S|,|T|)\in B_n}\beta_{(\lambda,|S|,|T|)}\\ &\times\Bigg[\sum_{\substack{(S,\mathscr{P})\in\mathscr{E}\\ \lambda_{\mathscr{P}}=\lambda,S\ni i}}\ \sum_{T\in\mathscr{P}\setminus\{S\}}|T|w(S,\mathscr{P})-\sum_{\substack{(S,\mathscr{P})\in\mathscr{E}\\ \lambda_{\mathscr{P}}=\lambda,S\not\ni i\\ |\mathscr{P}(i)|=|T|}}|S|w(S,\mathscr{P})\Bigg]\end{aligned}$$

$\beta_{(\lambda,|S|,|T|)}$'s satisfying conditions (4.5) and (4.6) for $(\lambda,\delta)\in F_n$) for every game w and for every player i, then every $\beta_{(\lambda,|S|,|T|)}$ vanish.

Thus, for given $(\lambda,|S|,|T|)\in B_n$ let $S=\{1,\ldots,s\}$ and $\mathscr{P}$ be any partition such that $S\in\mathscr{P}$ and $\lambda_{\mathscr{P}}=\lambda$. Also let $T\in\mathscr{P}$. Let $w=u_{(S,\mathscr{P})}$ and pick any $i\in T$. Then the above sum reduces to

$$0=\beta_{(\lambda,|S|,|T|)}$$ ■

Example 4.5 *For $n=4$, every LSEN solution takes the form given in (3.6), where the β's satisfy:*

$$-\beta_{([1111],1,1)}+2\beta_{([211],2,1)}-2\beta_{([211],1,2)}=0$$
$$2\beta_{([22],2,2)}-\beta_{([31],1,3)}-\beta_{([211],1,1)}=0$$
$$-2\beta_{([211],2,1)}+\beta_{([31],3,1)}-2\beta_{([22],2,2)}=0$$

which are the equations derived from the nullity restrictions in the expression of player 1's payoff. The table 3.2 associated with these values shows the weights corresponding to *payment allocation solutions for different procedures, such as Shapley, Myerson, among others. Each column in the table represents a particular set of parameters β's associated with a specific solution.*

The key observation is that the equations are satisfied only for the solutions ψ^{PN} and ψ^{MPW}, in addition to the Myerson value. This indicates that these particular payment allocation procedures meet the constraints established by the system of equations.

Remark 1 *In (Hernández-Lamoneada et al. [7]), it is proved that all linear, symmetric and efficient solutions can be uniquely expressed by (4.4) for arbitrary real numbers $\{\beta_{(\lambda,|S|,|T|)} \mid (\lambda,|S|,|T|) \in B_n\}$. Denoting by $LSE\left(\widetilde{\Gamma}^{(n)}\right)$ the vector space of all linear, symmetric and efficient solutions on $\widetilde{\Gamma}$ in n players, it is also pointed out that* $\dim LSE\left(\widetilde{\Gamma}^{(n)}\right) = |B_n|$.

Corollary 4.1.1 *The space of all linear, symmetric, efficient and null solutions in n players has dimension $|B_n| - |F_n| - 1$.*

We will denote by $LSEN\left(\widetilde{\Gamma}^{(n)}\right)$ the vector space of linear, symmetric, efficient and null solutions on $\widetilde{\Gamma}$ in n players.

Remark 2 *The relation*

$$\dim LSE\left(\widetilde{\Gamma}^{(n)}\right) = \dim LSEN\left(\widetilde{\Gamma}^{(n+1)}\right)$$

holds for every n. It is not clear what is the meaning of the fact that there are as many linear, symmetric and efficient solutions in n players and linear symmetric, efficient and null solutions in $n+1$ players.

We compute some cases for the dimension of families of solutions:

n	$\dim\widetilde{\Gamma}^{(n)}$	$\dim LSE\left(\widetilde{\Gamma}^{(n)}\right)$	$\dim LSEN\left(\widetilde{\Gamma}^{(n)}\right)$
2	3	1	0
3	10	3	1
4	37	7	3
5	151	14	7
6	674	26	14

Remark *Now, we can define the marginal contribution of a player i in the context of the partition function $w \in \tilde{\Gamma}$ as follows: Given a partition function $w \in \tilde{\Gamma}$ and a coalition $S \subseteq N$, the marginal contribution of player i in the coalition S is defined as:*

$$MC_i(S,\mathscr{P}) = w(S,\mathscr{P}) - w(S^{-i},\{(\mathscr{P}(i))^{-i},\{i\}\} \cup \mathscr{P} \setminus \{\mathscr{P}(i)\}) \tag{4.11}$$

The marginal contribution of a player $i \in N$ to $(S,\mathscr{P}) \in \mathscr{E}$ can be understood in 2 aspects:

1. *Direct* ($i \in S$, i.e., $S = \mathscr{P}(i)$):

$$MC_i(S,\mathscr{P}) = w(S,\mathscr{P}) - w((\mathscr{P}(i))^{-i},\{(\mathscr{P}(i))^{-i},\{i\}\} \cup \mathscr{P} \setminus \{\mathscr{P}(i)\})$$

2. *Indirect* ($i \notin S$):

$$MC_i(S,\mathscr{P}) = w(S,\mathscr{P}) - w(S,\{(\mathscr{P}(i))^{-i},\{i\}\} \cup \mathscr{P} \setminus \{\mathscr{P}(i)\})$$

The direct marginal contribution evaluates how the presence of player i in the coalition S affects the distribution of wealth in that specific coalition. It compares the original distribution of wealth in S with the adjusted distribution that would result if i decided to leave the coalition and work alone. If the difference is positive, the presence of i improves the distribution of wealth in S; if negative, it could be negatively affecting that distribution. In summary, MC_i *quantifies the direct impact of player i participating in the wealth of coalition S.*

In the case of indirect marginal contribution, if i is not in S, we want to evaluate how the situation would be affected if i decided to join. To do this, we consider how the total benefit of the team in S changes when i leaves another team, $T \in \mathscr{P}$ *(which is not S), and decides to work alone. This indirect marginal contribution helps us understand the potential impact of i's participation in S by considering their exit from another team.*

In essence, we are exploring how the presence or absence of i can influence the group's benefits, either directly when they are already part of the team or indirectly when evaluating how their decision to join would affect other teams. The structure of the partition is adjusted to isolate i and evaluate this impact more clearly.

So, the marginal contribution represents the total wealth generated by the coalition S that is distributed among the players according to the partition function w. The marginal contribution of i in S is calculated by taking the difference between the original distribution and the adjusted distribution when i joins or leaves the coalition.

This quantity reflects the specific influence of player i on the wealth distribution in coalition S. If i has no influence (because he is a null player), the marginal contribution will always be zero, indicating that their presence or absence does not affect the wealth distribution in that coalition.

In this context, it is important to highlight that it is possible to express *LSEN* (Linear, Symmetric, Efficient, and Null) solutions in partition function games as a linear combination of marginal contributions. By focusing on the payment of player one ($\psi_1(w)$), we have shown that this representation provides a clearer and more accessible insight into how the individual actions of each player can marginally affect the final outcome of the game.

Example 4.6

$$\begin{aligned}
\psi_1(w) \quad = \quad & 2\beta[w(\{1\},\{\{1\},\{2\},\{3\}\}) - w(\emptyset,\mathscr{P})]+ && \textit{Direct} \\
& (\frac{2}{6}-\beta)[w(\{1\},\{\{1\},\{2,3\}\}) - w(\emptyset,\mathscr{P})]+ && \textit{Direct} \\
& \frac{1}{6}[w(\{1,2\},\{\{1,2\},\{3\}\}) - w(\{2\},\{\{1\},\{2\},\{3\}\}]+ && \textit{Direct} \\
& \frac{1}{6}[w(\{1,3\},\{\{1,3\},\{2\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\}]+ && \textit{Direct} \\
& (\beta-\frac{1}{6})[w(\{2\},\{\{1,3\},\{2\}\}) - w(\{2\},\{\{1\},\{2\},\{3\}\}]+ && \textit{Indirect} \\
& (\beta-\frac{1}{6})[w(\{3\},\{\{1,2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\}]+ && \textit{Indirect} \\
& \frac{1}{3}[w(\{1,2,3\},\{\{1,2,3\}\}) - w(\{2,3\},\{\{1\},\{2,3\}\}] && \textit{Direct}
\end{aligned}$$

The expression $\psi_1(w)$ represents the specific payment assigned to player 1, calculated from a partition function w. Each term in the formula reflects the contribution of player 1 in different scenarios, considering the coalitions in which they participate. The classification as *Direct* or *Indirect* indicates whether the contribution is direct when present in the coalition or indirect if not.

4.2.1 Null Players Definitions

Now we turn to different versions of players who do not contribute to the game, commonly referred to as null players. There are various iterations of null players, and each version gives rise to a different extension of the Shapley value.

In the Shapley context, given a coalition T, if $i \notin T$, then i is considered null. However, this is not necessarily the case with the Myerson carrier in $\tilde{\Gamma}$, as players may have an individual value but do not generate surplus when joining coalitions.

Example 4.7 *In the context of the Shapley carrier T, if a player i does not belong to T, then it is considered null. However, this condition does not necessarily hold in the Myerson carrier, as we will see below.*

i) Shapley: *From Definition (3.2), $T = \{1\}$ is the carrier of $\upsilon \in \Gamma$ if $\forall S \subseteq N$*

$$\begin{array}{lll}
\upsilon(\{2\}) = 0 & \upsilon(\{3\}) = 0 & \upsilon(\{2,3\}) = 0 \\
\upsilon(\{1,2\}) = \upsilon(\{1\}) & \upsilon(\{1,3\}) = \upsilon(\{1\}) & \upsilon(\{1,2,3\}) = \upsilon(\{1\})
\end{array}$$

and $w(\{1\})$ arbitrary.

According to Definition (3.3), players 2 and 3 are classified as null in $\upsilon \in \Gamma$ if for any subset $S \subseteq N$.

$$\begin{array}{lll}
\upsilon(\{2\}) = 0 & \upsilon(\{3\}) = 0 & \upsilon(\{2,3\}) = 0 \\
\upsilon(\{1,2\}) = \upsilon(\{1\}) & \upsilon(\{1,3\}) = \upsilon(\{1\}) & \upsilon(\{1,2,3\}) = \upsilon(\{1\})
\end{array}$$

Both definitions give us the same conditions; therefore, it is natural to obtain the same Shapley value, i.e., if a player does not belong to the carrier set, then he is a null player.

For instance, let's consider the following game, where $\{1\}$ is a carrier:

$$\begin{array}{lll} \upsilon(\{1\})=5 & \upsilon(\{1,2\})=5 & \\ \upsilon(\{2\})=0 & \upsilon(\{1,3\})=5 & \upsilon(\{1,2,3\})=5 \\ \upsilon(\{3\})=0 & \upsilon(\{2,3\})=0 & \end{array}$$

The Shapley value for this game is:

$$Sh(\upsilon)=(5,0,0)$$

ii) Myerson: *According to definition (3.9), $T=\{1\}$ is a carrier set of $w \in \tilde{\Gamma}$ if $\forall (S,\mathscr{P}) \in \mathscr{E}$.*

$$\begin{array}{llll} w(\{2\},\{\{1\},\{2\},\{3\}\}) & = w(\{3\},\{\{1\},\{2\},\{3\}\}) & = 0 & \\ w(\{2\},\{\{1,3\},\{2\}\}) & = w(\{3\},\{\{1,2\},\{3\}\}) & = w(\{2,3\},\{\{1\},\{2,3\}\}) & = 0 \\ w(\{1,2\},\{\{1,2\},\{3\}\}) & = w(\{1,3\},\{\{1,3\},\{2\}\}) & = \mathbf{w(\{1\},\{\{1\},\{2\},\{3\}\})} & \\ w(\{1,2,3\},\{\{1,2,3\}\}) & = \mathbf{w(\{1\},\{\{1\},\{2,3\}\})} & & \end{array}$$

with $w(\{1\},\{\{1\},\{2\},\{3\}\})$ and $w(\{1\},\{\{1\},\{2,3\}\})$ arbitrary.

According to our definition (4.1) of a null player, players 2 and 3 are null in $w \in \tilde{\Gamma}$ if $\forall (S,\mathscr{P}) \in \mathscr{E}$.

$$\begin{array}{llll} w(\{2\},\{\{1\},\{2\},\{3\}\}) & = w(\{3\},\{\{1\},\{2\},\{3\}\}) & = 0 & \\ w(\{2\},\{\{1,3\},\{2\}\}) & = w(\{3\},\{\{1,2\},\{3\}\}) & = w(\{2,3\},\{\{1\},\{2,3\}\}) & = 0 \\ w(\{1,2\},\{\{1,2\},\{3\}\}) & = w(\{1,3\},\{\{1,3\},\{2\}\}) & = \mathbf{w(\{1\},\{\{1\},\{2\},\{3\}\})} & \\ w(\{1,2,3\},\{\{1,2,3\}\}) & = \mathbf{w(\{1\},\{\{1\},\{2\},\{3\}\})} & & \end{array}$$

with $w(\{1\},\{\{1\},\{2\},\{3\}\})$ arbitrary.

Note that our definition of a null player and Myerson's carrier definition reveal that if a player is not in T, it does not necessarily mean he is null. This will be better visualized in the following example.

Let's consider the following game,

$\mathscr{P}$	**Coalitions**	**Worth**
1	{1}, {2}, {3}	(5,0,0)
2	{1,2}, {3}	(5,0)
3	{1,3}, {2}	(5,0)
4	{2,3}, {1}	(0,7)
5	{1,2,3}	(7)

The payoff for players, from the solution $\psi(w)$ that satisfies the axioms of linearity, symmetry, efficiency and nullity, is

$$\psi(w)=(\frac{19}{3}-4\beta,\frac{1}{3}+2\beta,\frac{1}{3}+2\beta) \tag{E1}$$

The following graph shows this family of solutions:

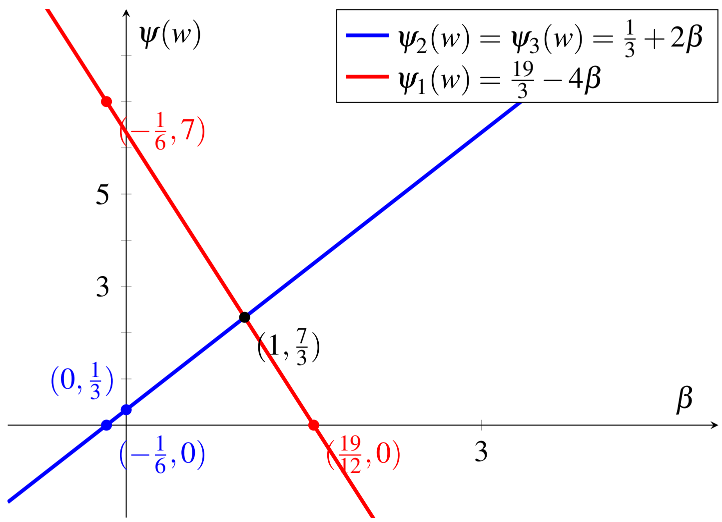


Figure 4.1 Family of linear, symmetric, efficient, and null solutions for 3 players. Value (E1).

In this context, the payments along the lines represent the payment assignments for the players as a function of the parameter β. Each line in the graph corresponds to the payments of a specific player, and its slope determines how those payments change as we adjust the value of β.

1. Blue Lines (Players 2 and 3): Both blue lines are parallel, indicating that players 2 and 3 receive the same payment amount. The slope of these lines is positive (2), indicating that the payments of players 2 and 3 increase as β increases. When $\beta = 0$, both players receive a payment of $\frac{1}{3}$, and this payment increases linearly with β.
2. Red Line (Player 1): The red line has a negative slope (-4), meaning that the payment of player 1 decreases as β increases. When $\beta = 0$, player 1 receives a payment of $\frac{19}{3}$, and this payment decreases linearly with β.

The intersection points with the axes indicate the payments when β takes specific values. For example, the point $(0, \frac{19}{3})$ on the red line indicates the payment of player 1 when $\beta = 0$, and the point $(-\frac{1}{6}, 0)$ on the blue lines shows the common payment of players 2 and 3 when $\beta = -\frac{1}{6}$.

In general, moving along these lines in the graph reflects how the payment assignments for each player evolve consistently in response to changes in the parameter β according to the solution given by equation (E1), causing the multiplicity of LSEN solutions.

According to the above, we are dealing with null players in various forms, so we refer to the corresponding axioms as null player axioms [1].

Definition 4.2 (Null Player in Partition Function Form Games, Pham Do and Norde [8]) *Given a partition function form game* (N,w)*, a player* i *is a null player if for all* $\mathscr{P} \ni \{i\}$*,*

- $w(\{i\},\mathscr{P})=0$ *and*

- $w(T^{+i}, \mathscr{P}^i(T)) = w(T, \mathscr{P}) \quad \forall T \in \mathscr{P}$

The first part of the definition states that null players have no value by themselves and cannot create value by joining a coalition, that is, their marginal contribution is also zero. This property does not say anything about the way such players affect other coalitions.

Our Definition:

- It focuses on the redistribution of wealth in the game when the null player is present or absent in a coalition.
- It evaluates how the partition $\mathscr{P}$ changes when the null player is involved in a coalition and when they are not.
- It does not establish specific conditions on the value of the null player or their impact on other coalitions.

Definition from (Pham Do and Norde, 2007):

- It is based on two distinct conditions:
  1. The null player has no value by himself and does not contribute to any coalition they participate in.
  2. The wealth generated in a coalition before and after the null player leaves their coalition and joins another is equal.
- Does not consider the impact of the null player on the redistribution of wealth between coalitions but rather focuses on their nullity in terms of contribution and maintaining constant wealth in the coalitions, without considering their impact on redistribution.

In summary, the main difference lies in the focus of each definition: our proposal focuses on how the null player affects the distribution of wealth, while the definition from the literature focuses on the nullity of the player in terms of contribution and on maintaining constant wealth in the coalitions, without considering their impact on redistribution.

Definition 4.3 (Dummy player, Macho-Stadler [11]) *Given a partition function form game (N,w), a player i is a null player if for all $\mathscr{P} \ni \{i\}$,*

$$w(T^{+i}, \mathscr{P}^i(T)) = w(T, \mathscr{P}) \quad \forall T \in \mathscr{P}$$

Note that Pham Do and Norde require the wealth associated with the coalition $\{i\} \in \mathscr{P}$ to be zero for a player to be considered null, while Macho-Stadler does not include this additional condition. Both definitions share the consistency condition of wealth in coalitions that involve the null player.

Definition 4.4 (Efficient-Cover Null Player, Hafalir [12]) *Given a partition function form game (N,w), a player i is an efficient-cover null player if for all $S \subseteq N$, the efficient-cover function $\overline{w}$ satisfies*

$$\overline{w}(S, \{S\} \cup [\overline{S}]) = \overline{w}(S^{-i}, \{S^{-i}\} \cup [\overline{S}]^{+i})$$

Our Definition:

- It directly applies to a partition function in a game $(N, \mathscr{P})$.

- The comparison focuses on how wealth is distributed across different coalitions with or without the presence of player i.
- It does not make explicit assumptions about the structure of the remaining players in the coalitions.

Hafalir's Definition (Efficient-Cover Null Player):

- It applies to the efficient partition of a game and its partition function $\bar{w}$.
- The comparison is made in terms of how the efficient partition behaves rather than the original partition.
- It explicitly assumes that the remaining players in the coalition are singletons (individual players without coalitions).

In summary, the main difference lies in the focus and the context of application. Our proposal centers on the original partition function and applies more generally to different coalition structures, without making assumptions about the remaining players. In contrast, Hafalir's definition focuses on the efficient partition and assumes a specific structure of the remaining players (all are singletons) in the coalitions.

Definition 4.5 (Null Player in the Strong Sense, Bolger [13]) *Given a partition function form game (N,w), a player i is a null player in the strong sense if for all $(S,\mathscr{P}) \in \mathscr{E}$ and $T \in \mathscr{P} \cup \{/\emptyset\}$ where $T \neq S$,*

$$w(S,\mathscr{P}) = w(S^{-i},\mathscr{P}^{i}(T))$$

Therefore, the value of a coalition is not changed if a null player in the strong sense is transferred to another coalition in the partition. As a special case, looking at $S = \{i\}$ shows that the value of a singleton that is a null player in the strong sense is zero in any coalition structure and, therefore, the term null is more appropriate. Notice also that $i \in S$ is not required. In essence, this property captures the irrelevance of null players (Macho-Stadler et al. [15]) in two aspects:

1. A null player in the strong sense does not contribute to a coalition since if $i \in S$, then the departure of i does not change the payoff of S. (Using the relation on T^{+i} also shows that i does not increase the payoff of T by joining.)
2. A null player in the strong sense does not generate externalities since moving i between coalitions does not change the payoff of third parties, that is, coalitions $S \neq \{i\}$.

The concept of a null player in the strong sense is intimately related to extended carriers. If $C \subseteq N$ and C^{-i}, where $i \in C$, are both extended carriers, then i is a null player in the strong sense (McQuillin [14]).

Now,

- **Nullity Condition:** In our proposal, a player i is considered a null player if, for every pair $(S,\mathscr{P})$ in the efficient coalition structure, the wealth generated by the coalition S remains the same, regardless of whether i is present or absent in that coalition. In Bolger's definition, a player i is a null player in the strong sense if, for every pair

$(S,\mathscr{P})$ and for every set T in $\mathscr{P}$ (including the possibility of transferring i to other coalitions), the wealth of S is invariant. This implies that a strong null player not only doesn't affect the coalition to which they belong but also does not affect other coalitions to which they might be transferred.

- **Contribution to Coalitions:** In our proposal, it is emphasized that a null player does not contribute to a coalition since their presence or absence does not affect the wealth distribution. In Bolger's definition, this is reflected in the property that the exit of a strong null player (if $i \in S$) does not change the outcome of coalition S, indicating that this player does not contribute to the coalition in terms of wealth.
- **Generation of Externalities:** Our proposal does not specifically mention externalities, but Bolger's definition addresses this aspect by stating that a strong null player does not generate externalities since moving i between coalitions does not affect the rewards of third parties (coalitions $S \neq \{i\}$).
- **Presence of Player i:** In our proposal, it is not required for i to be present in coalition S to be considered a null player. Bolger's definition does not mention the need for i to be present in S but focuses on how their transfer affects rewards.

Definition 4.6 (Null Player with Steady Marginality, Skibski [16]) *A player i is a null player with steady marginality if for all partitions $\mathscr{P} \in \tilde{\Pi}(N)$,*

$$\sum_{\substack{T \in \mathscr{P} \\ T \not\ni i}} w(\mathscr{P}(i),\mathscr{P}) - w(\mathscr{P}(i)^{-i},\mathscr{P}^{i}(T)) = 0 \quad \textit{and} \tag{4.12}$$

$$w(N,\{N\}) - w(N^{-i},\{N^{-i},\{i\}\}) = 0 \tag{4.13}$$

The definition of *Null Player with Steady Marginality* by Skibski establishes a special condition for a player i in a cooperative game. Skibski (2011, [16]) claims that a player that is a null player in the strong sense is also a player with steady marginality, so, a player is said to be a *null player with steady marginality* if, for all ways of partitioning the players into coalitions, a specific condition holds.

This condition involves two parts:

1. The first part of the condition (4.12) states that, for each partition $\mathscr{P}$ that does not include player i in any coalition T, the sum of differences between two wealth values must be equal to zero. These wealth values are associated with player i participating or not participating in coalition $\mathscr{P}(i)$.
2. The second part of the condition states that the wealth difference between the total game N and the game excluding player i must also be equal to zero. This implies that player i's contribution to the total game and their contribution when excluding i along with a coalition containing only i must be equivalent.

In summary, this definition characterizes a null player with steady marginality as one whose contribution to wealth is consistent and balanced across all possible partitions of the player set.

1. **Difference in Formulation:** In our proposal, the condition for a player to be null is based on an equality that compares the wealth of a coalition with the wealth of

the same coalition after removing player i. In contrast, Skibski's definition is based on two separate conditions: one related to the sum of wealth differences between coalitions in a partition and another related to the wealth of the entire set of players.

2. **Steady Marginality:** Skibski's definition incorporates the concept of *steady marginality*, which implies that player i cannot join an empty coalition and form a new coalition. This restriction is not present in our proposal.
3. **Relationship between Definitions:** Skibski points out that a null player in the strong sense (satisfying the conditions of our definition) is also a player with constant marginality. However, the two definitions are not mutually related, indicating a difference in approach and requirements to be a null player.

Final Comments

In summary, this work has focused on the quest for alternative characterization of Myerson's value in partition function form games. Unlike Shapley, whose uniqueness is established in characteristic function games, we have demonstrated that in partition function form games, it is not possible to find a characterization that is simultaneously linear, symmetric, efficient, and null. This finding highlights the fundamental structural differences between these two types of cooperative games.

The introduction of a null player in the characterization reveals a family of parameterized solutions, indicating the diversity of possible outcomes and the lack of uniqueness in this context. However, to achieve a unique solution in partition function form games, it is suggested to explore additional axioms or consider more specific axioms that can constrain the parameterization.

One possible avenue to attain uniqueness could be the introduction of additional constraints on the alternative parameters, such as specific conditions on their values or relationships between them. These constraints could be derived from desirable additional properties in the specific problem context or from empirical analyses of concrete situations.

In conclusion, this work not only provides a clear insight into the limitations of a unique characterization in partition function form games but also suggests that the pursuit of uniqueness could benefit from the inclusion of additional constraints based on more detailed theoretical considerations or practical application of the model. These suggestions offer a valuable direction for future research, opening new opportunities to better understand the nature of solutions in cooperative games.

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